284. You are given 50 intervals on a line. Prove that at least one of the following statements about those intervals is true:

(a) There are 8 intervals all of which have at least one point in common.

(b) There are 8 intervals so that no two of them have a common point.

Solutions to Problems 261–272 (Vol. 11 No. 1)

261. In a right-angled triangle, the shortest side is a cm long, the longest side is c cm long and the other side is b cm. If a, b, c are all integers, when does $a^2 = b + c$?

Answer: By Pythagoras' Theorem, $a^2 = c^2 - b^2 = (c-b)(c+b)$. This equals $c + b$ if and only if $c - b = 1$ (i.e. when the hypotenuse is just one unit longer than the next longest side).

Thus

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (b+1)^2 - b^2 \\ &= 2b + 1 \end{aligned}$$

a cannot be even and so $a = 2d + 1$ for some integer d.

So

$$\begin{aligned} 2b + 1 &= (2d+1)^2 \\ &= 4d^2 + 4d + 1 \end{aligned}$$

i.e. $b = 2d^2 + 2d$ and $c = b + 1 = 2d^2 + 2d + 1$.

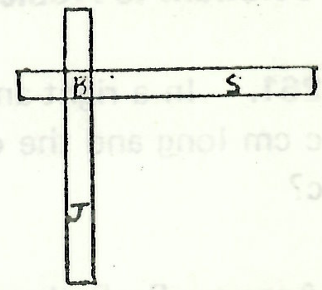
262. On his birthday in 1975 John reaches an age equal to the sum of the digits in the year he was born. What year was that?

Answer: The sum of the digits of any date earlier than 1975 does not exceed 27 (which occurs for the year 1899). Hence John is at most 27 years old and was born no earlier than 1948. The largest sum of digits of a date between 1948 and 1975 is 25 (the year 1969), and the smallest such sum is 15 (the year 1950); therefore John's age will lie between 15 and 25. His birthdate must have been in the fifties, say 1950 + x where x is a digit. His age will reach 25–x in 1975. Hence

$$25 - x = 15 + x, \quad \text{giving } x = 5.$$

John was born in 1955.

263. Three hundred soldiers are positioned in 15 rows each containing 20 soldiers. From each of the 20 columns thus formed the shortest soldier falls out and the tallest of these 20 men proves to be private Jones. They then resume their places on the parade ground. Next the tallest soldier in each row falls out, and the shortest of these 15 soldiers is private Smith. Who is taller, Jones or Smith?



Answer: Smith is taller than Jones. If they are in the same row, Smith is the tallest in that row. If they are in the same column, Jones is the shortest in that column. Otherwise there is some soldier (Brown, say) in the same row as Smith and the same column as Jones. Smith is taller than Brown, and Jones is shorter than Brown (see figure).

264. In the 1974 cricket XI there were 7 boys who had been in the 1973 XI, and in the 1973 XI there were 8 boys who had been in the 1972 XI. What is the least number who have been in all three XI's?

Answer the same question with x instead of 7 and y instead of 8. For what values of x and y is it possible that there were no boys in all three XI's?

Answer: One is expected here to make the rather unrealistic assumption that the cricket XI of any year comprised exactly 11 players. Then in the 1973 team there were only $11-8 = 3$ members who had not played in the 1972 team. Therefore of the 7 who were in both the 1973 and 1974 sides, at most 3 had not played in 1972, so at least $7-(11-8) = 4$ had been in all three teams.

Repeating the argument we see that at least $x + y - 11$ were in all three teams, or more realistically, $x + y - N$, where N is the number of players making up the 1973 cricket XI. Thus it is possible that there were no boys in all three teams for any values of x and y with $x + y$ at most N.

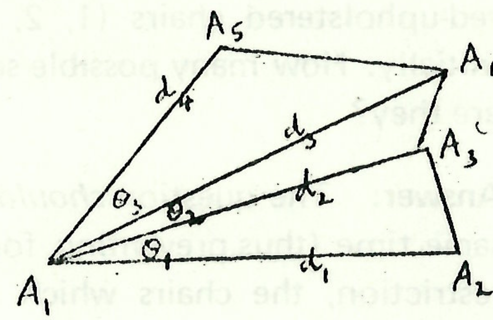
265. If you are required to make an exact copy of an irregular hexagon given a ruler and a protractor, what is the least number of measurements you would have to make? If you had no protractor could you still do it? If so would a greater number of measurements be needed?

What would be the least number of measurements required to copy an irregular polygon with n sides?

Answer: Suppose the polygon has n sides $A_1A_2, A_2A_3, \ldots A_nA_1$. In drawing this polygon we may take any point as A_1 and draw the side A_1A_2 in any direction from A_1 (by rotating the paper on which we are drawing the polygon). Consider now co-ordinate axes drawn with the origin at A_1 and the x-axis along the side A_1A_2. Clearly the position of A_2 then depends on the distance A_1A_2*. The remaining n−1 points A_i may be thought of as ordered pairs (x_i, y_i) and so each requires at least two measurements to determine its position in relation to A_1. Thus the total number of measurements for the polygon is at least

$$1 + 2(n-1) = 2n-3.$$

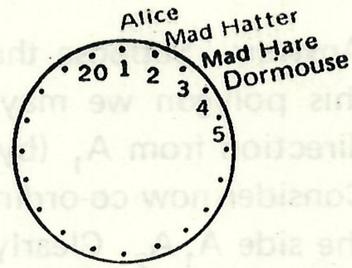
To see this number is enough, join each vertex A_i to A_1. Then A_2 may be determined (relative to A_1 and the direction of A_1A_2) using the distance A_1A_2*; A_3 may be determined from A_1 and A_2 using the distance A_1A_3* and the angle $\angle A_2A_1A_3$; A_4 may be determined from A_1 and A_3 using the distance A_1A_4* and the angle $\angle A_3A_1A_4$; etc.



All the measurements may be lengths (a triangle is determined up to congruence by the lengths of its sides, and any polygon can be dissected into triangles by drawing diagonals). Thus we can do without the protractor. However at least one length measurement is imperative since figures with corresponding equal angles need not be congruent, only similar.

Comment:— You may be inclined to suppose that if the protractor was not available the actul construction of the polygon could not be accomplished with ruler alone unless one "cheated" by using it in a non-standard way to draw circles; viz. keep one mark on the ruler on the centre of the circle, and hold the pencil on a second mark while rotating the ruler (This makes very wobbly circles if you care to try it. Not really a satisfactory alternative to compasses.) This, however is not true. You might care to try your hand at showing how a triangle whose sides are of known length can be accurately constructed with ruler and pencil. (Allowable operations:— (1) Rule a line through up to two given or previously constructed points. (2) Mark off a segment of any given length on a ruled line, either end-point being given. (3) Measure the length of any line segment.)

266. At the mad hatter's afternoon tea party there are twenty seats, 4 neighbouring ones with red cushions (1, 2, 3 and 4 in the diagram) being initially occupied by Alice, the mad hatter, the march hare and the dormouse. Instead of all moving round one seat at a time (as in the classical story) the members of the party move quite independently as the fancy takes them, but always to an unoccupied seat 7 places away in either direction. Even the dormouse proves to be wakeful enough to carry out this complicated manoeuvre several times. At a later time it turns out that they are again sitting next to one another on the same red-upholstered chairs (1, 2, 3 and 4), though none is in the same place as initially. How many possible seating arrangements are there at the finish and what are they?



Answer: The question *should* have stated that no two people could move at the same time (thus preventing, for example, two of them swapping seats). Under this restriction, the chairs which Alice is able to visit by the given manoeuvre (in order) are:

1, 8, 15, 2, 9, 16, 3, 10, 17, 4,
11, 18, 5, 12, 19, 6, 13, 7, 14, 1,
8, . . .

Since this list includes all the chairs, any of the four at the tea party may end up on any chair by the same manoeuvre. However, their relative positions in the above cycle cannot change (e.g. if the Mad Hatter is on seat 9 and Alice is on seat 2, she cannot pass the Mad Hatter since she must move to an unoccupied chair and so can only move back to seat 15). Hence there are only 4 ways in which the seats 1, 2, 3, 4 can be occupied, viz.

1 Alice	2 Mad Hatter	3 March Hare	4 Dormouse	(As initially)
2 Alice	3 Mad Hatter	4 March Hare	1 Dormouse	
3 Alice	4 Mad Hatter	1 March Hare	2 Dormouse	
4 Alice	1 Mad Hatter	2 March Hare	3 Dormouse	

The first of these has been specifically ruled out.

Note: If two people were allowed to move at the same time, the number of final arrangements would simply be the number of ways of arranging the four people with none of them in their original position viz. 9. You might like to try the general problem and let us know what the answer is when n people are re-arranged so that none of them is in their original position.

267. A chain has 2047 links in it. It is to be separated into a number of pieces by cutting and disengaging appropriate links, in such a way that any number of links (from 1 to 2047) may be gathered together from the parts of chain thus produced. What is the smallest number of links which must be cut to achieve this?

(For example, if the chain had 7 links it would have sufficed to disengage 1 link, the third from an end, producing pieces of chain with 1 link, 2 links and 4 links. You can easily check that any number of links up to 7 can be gathered using these pieces.)

Answer: By cutting and disengaging 7 links of the chain a further eight lengths containing respectively 8, 16, 32, 64, 128, 256, 512, and 1024 links may be left. It is now obvious that using the 7 separately disengaged links we can make up any number up to 7; using also the 8 link piece we can make up any number up to 15; and so on. Eventually any number of links up to 2047 can be gathered using these pieces.

Note: This solution depends on writing a number in binary notation.

268. Prove that $11^{10}-1$ is divisible by 100.

Answer: $11^2 = 100 + 21$.

So $11^4 = 100^2 + 2.21.100 + 441 = 100k + 41$ where k is an integer

and $11^5 = 1100k + 451 = 100m + 51$ where m is an integer.

Thus $11^{10} = (100m + 51)^2 = 10000m^2 + 2.100m.51 + 2601$

$= 100n + 1$ where n is an integer.

Q.E.D.

269. (i) Show that for any positive integer n

$$2 < (1+\tfrac{1}{n})^n < 3.$$

(ii) Which is larger, 1000^{1000} or 1001^{999}?

Answer: *(i)* By the binomial theorem.

$$(1+\tfrac{1}{n})^n = 1 + {}^nC_1 \tfrac{1}{n} + {}^nC_2 (\tfrac{1}{n})^2 + \ldots + {}^nC_n (\tfrac{1}{n})^n$$

$$= 1 + \frac{n}{1}\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{n^3} + \ldots + \frac{n(n-1)\ldots 3.2.1}{1.2.3 \ldots n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1(1-1/n)}{2} + \frac{1(1-1/n)(1-2/n)}{2.3} + \ldots + \frac{1(1-1/n)(1-2/n)\ldots(1-(n-1)/n)}{n!}$$

Therefore $2 < (1+\frac{1}{n})^n < 2 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} < 3.$

(ii) By *(i)*, $(1 + \frac{1}{1000})^{1000} < 3$. From this

$$1000^{1000} > \tfrac{1}{3}(1001)^{1000} = \frac{1001}{3}(1001)^{999}$$

That is, 1000^{1000} is over 300 times larger than 1001^{999}.

270. Find all positive integers between 1 and 100 having the property that $(n-1)!$ is not divisible by n^2.

Answer: By trying the integers between 1 and 100 using a calculator, you will see that n^2 does not divide $(n-1)!$ for all n between 1 and 11; and for n between 10 and 100, n^2 does not divide $(n-1)!$ if and only if n is a prime number or twice a prime number.

This suggests that, for all n greater than 10, the only n such that n^2 does not divide $(n-1)!$ are n = prime number or twice a prime number, which we will now prove. First, for n = p (a prime), $(n-1)! = 1.2.3 \ldots (p-1)$ which is not divisible by p^2 (not even by p). Also, if n = 2p, $(n-1)! = 1.2. \ldots p \ldots (2p-1)$ which is not divisible by p^2, and therefore not by n^2.

If n = ab where a and b are relatively prime and both greater than 2, the integers a, 2a, b and 2b all occur as factors in $(n-1)!$ so n^2 divides $(n-1)!$ This leaves for consideration only integers $n = p^k$, 2^k and $n = 2p^k$ where p is an odd prime, k a positive integer.

Now the largest power of 2 which divides $(2^k)!$ is 2^c where $c = 2^k - 1$. (For example, 8! is divisible by 2^7.) To prove this, observe that every second factor of $1.2.3.4. \ldots 2^k$ is even, and removing one factor of 2 from each of these 2^{k-1} terms contributes 2^{k-1} factors of 2. Every 4th factor is divisible by 4, so a further 2^{k-2} factors of 2 are obtained by taking one more factor from each of these.

Similarly we can find an extra 2^{k-3} factors of 2 from the multiples of 8, and so on, yielding eventually a total of $2^{k-1} + 2^{k-2} + \ldots + 2 + 1 = 2^k - 1$ factors of 2, as claimed above. Omitting the last factor, 2^k, we see that if $n = 2^k$, $(n-1)!$ is divisible by 2^m where $m = 2^k - 1 - k$. For such n, n^2 divides $(n-1)!$ provided $2k \leqslant 2^k - 1 - k$, i.e. $3k + 1 \leqslant 2^k$ which is satisfied if $k \geqslant 4$ but not for k = 1, 2 or 3.

A similar (but simpler) analysis shows that the largest power of p (an odd prime) which divides $(p^2-1)!$ is p^{p-1}. Hence if $n = p^2$, $n^2 = p^4$ divides $(n-1)!$ provided $p - 1 \geqslant 4$. If $n = 2p^2$ then p^{2p} divides $(n-1)!$, and as $4 < 2p$ for any odd prime p, n^2 always divides $(n-1)!$

The same result is easily found for any $n = p^k$, $k > 2$ or $n = 2p^k$, $k > 2$.

271. Prove that if the sum of the fractions $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ (where n is a positive integer) is put in decimal form, it forms a non-terminating decimal which is periodic after several terms.
(e.g. For n = 3, $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} = .78\dot{3}$ is periodic after 2 decimal places.)

Answer: Let r/s be a fraction in lowest terms i.e. r and s are integers with no common factor. On converting to decimal form:—

(1) the decimal terminates if and only if $s = 2^a 5^b$ (a,b non negative integers);

(2) otherwise a periodic decimal results;

(3) the decimal is a *pure* recurring decimal (i.e. the period starts right from the first digit) if and only if neither 2 nor 5 is a factor of s. If $s = 2^a 5^b q$ where q and 10 are relatively prime, the number of digits before the decimal starts to recur is equal to the larger of a and b.

For proofs of these standard theorems on decimal expansions, see a textbook (or ask your teacher!).

Now $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} = \frac{3n(n+2) + 2}{n(n+1)(n+2)} = \frac{r}{s}$ after reduction to lowest terms. Exactly one of n, n+1 and n+2 is divisible by 3, but the numerator $3n(n+2) + 2$ is clearly not divisible by 3. Hence s contains the factor 3. By (i) above, the decimal expansion does not terminate, but by (2) it is periodic.

If n is odd, so is n+2 and hence $3n(n+2) + 2$ is odd. Since $n(n+1)(n+2)$ is even, 2 is a factor of s. If n is even, $3n(n+2)$ is divisible by 4 and so $3n(n+2) + 2$ is even but not divisible by 4. Since 4 is a factor of $n(n+1)(n+2)$ when n is even, we see that 2 is again a factor of s.

Since 2 must be a factor of s for all n, (3) tells us that the recurring decimal expansion of r/s is not pure, but has at least one digit before the period commences.

272. Let m and n be two relatively prime positive integers. Prove that if the $m + n - 2$ fractions

$$\frac{m+n}{m}, \frac{2(m+n)}{m}, \frac{3(m+n)}{m}, \ldots, \frac{(m-1)(m+n)}{m},$$

$$\frac{m+n}{n}, \frac{2(m+n)}{n}, \frac{3(m+n)}{n}, \ldots, \frac{(n-1)(m+n)}{n},$$

are plotted as points on the real number line, exactly one of these fractions lies inside each of the unit intervals (1,2), (2,3), (3,4), . . . , $(m + n-2, m + n-1)$. (e.g. If $m = 3$, $n = 4$, then 7/4 is between 1 and 2, 7/3 is between 2 and 3, 14/4 is between 3 and 4, 14/3 is between 4 and 5, and 21/4 is between 5 and 6.)

Answer: To make work easier, let us suppose that $m < n$. None of the given numbers is an integer since m and n are relatively prime. Clearly the smallest number in either list is greater than 1, and the largest is $(n-1)(m + n)/n = m + n - (m + n)/n = m + n-1-m/n$. So every number lies between 1 and $m + n-1$ and so in one of the given intervals. As there are the same number of intervals as numbers, to prove that each interval contains exactly one number, it is sufficient to show that no interval contains more than one.

The distance between neighbouring numbers in the first list is $(m + n)/m > 1$, so no two of these can both lie in the same interval $(k,k+1)$. A similar observation applies to the second list. It only remains to show that no number $r(m + n)/m$ is in the same interval as any number $s(m + n)/n$. This follows from the observation that the integer $(r + s)$ lies between these two numbers. Indeed $(r + s) - r(m + n)/m = (sm-rn)/m$ and $s(m + n)/n - (r + s) = (sm-rn)/n$.

Hence these two differences both have the same sign, and this completes the proof.

Successful Solvers of Problems 261-272

V. Arvanitis (Smiths Hill Girls' High) 261, 268.
M. Cook (James Ruse Ag. High) 262.
A. Fekete (Sydney Grammar) 266, 268, 269, 270, 271.
M. Hartley (Castle Hill High) 268, 269.
J. Keith (James Ruse Ag. High) 262.
G. Middleton (Marist Brothers, North Sydney) 264, 267, 271.
M. Nichols (C.E.G.G.S., Canberra) 261, 262, 263, 264, 265.
J. Rogers (Knox Grammar) 268, 269, 270, 271.
T. Rugless (St. Catherine's, Waverley) 262, 263.

Late Solvers of Problems 251-260

J. Burnett (James Ruse Ag. High) 256, 259.
M. Durie (Canberra Grammar) 256, 257, 258.
R. Kent (James Ruse Ag. High) 251, 252, 253.
J. Reeves (Wesley College) 256, 257, 259.
M. Reynolds (Marist Brothers, Pagewood) 253, 255, 259.

SUM MORE WIZARDS

In the last article, I told you of Zarah Colburn, who could multiply numbers like 8,476 and 2,459 together in his head in just a few seconds. Well, the German Johann Dase (1824-1861) was even better.

In about 1840 he multiplied in his head correctly, two 20 digit numbers in about 6 minutes; to find the product of two 40 digit numbers he took about 40 minutes; to find the product of two 100 digit numbers he only took about 9 hours. Once he found the square root of a 100-digit number in 52 minutes. But one day he suffered from a headache and got all of his sums wrong!!

Unlike Colburn, who displayed his powers at fairs, and Bidder, who simply enjoyed his great ability, Dase was employed by the Austrian Government, the Hamburg Academy of Science, the great Gauss himself, on projects involving huge calculations, such as the preparation of logarithm tables and tables of factors. In 1857 he offered to tabulate the factors of all numbers from 7,000,000 to 10,000,000 but he died with the project half complete.

It has been said that Gauss' payment of Dase to calculate for him is the first example of payment for computer time.

I wish to point out that none of the three men discussed was a Mathematician. They could calculate enormous numbers rapidly, but they did not discover anything new in Mathematics.

I will now mention several Mathematicians who exhibited phenomenal ability as children.

Blaise Pascal, 1623-1662 wrote an essay on sound, when he was 12. At 13, untutored in Geometry, he proved the "angle sum of a triangle" Theorem, using his own axioms and definitions. At 16 he wrote an essay on Conics, and at 18 invented a calculating machine.

Leonard Euler (1707-1783) received his Master of Arts degree when he was 17.

Joseph-Louis Lagrange (1736-1813) became Professor of Mathematics at the Royal School of Artillary at Turin at the age of sixteen.

Carl Frederick Gauss (1770-1855) pointed out an error in his fathers' accounts when he was three. When he was 10, without any instructions, he quickly found the answer to this problem:

$$81297 + 81495 + 81693 + \ldots + 100899 \text{(Arithmetic series)}$$

William Rowan Hamilton (1805-1865) could read English when he was three; at four he was interested in Geography, and was reading Latin, Greek and Hebrew. By the time he was 13, he could speak 13 languages.

All of these men are extremely important in the story of Mathematics, and I hope you enjoy reading about them.

References: "The World of Mathematics", Vol. 1, Newman; "Men of Mathematics", Vol. I, II, Bell.

SOME CORRECTIONS TO LAST ISSUE

The examples of speed in mental arithmetic in the last Parabola (page 27) contain a misprint and a flaw. The misprint is evident to those who know that .142857142857. . . is the decimal expansion of 1/7. The flaw comes in the calculation of the interest on £11111 for 11111 days at 5%. This was done by evaluating 11111 x 11111 x 5/36500, ignoring the fact that 30 consecutive years contain either 6 or 7 leap years. The answer thus depends on the day the interest started to accumulate and should be about £10 less.

Experiment shows that such problems can be carried out readily using a modern pocket calculator in much the time as the young genius took without one.

Moral: Never accept anything that you can readily check.

Peter Donovan

☆ ☆ ☆

Answer to Bookworm Quickie: 1 cm. (If you cannot see why, ask your librarian.)

Answer to Metallic Cryptarium:

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   94 003 +
   82 599
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  176 602
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