FOLLOW UP TO PYTHAGOREAN PATTERNS

In Vol. 11 No. 1, W.J. Ryan asked several questions about Pythagorean triples, which I will try to answer.

A Pythagorean triple is an ordered triple (a,b,c) of positive integers such that $0 < a < b < c$ and $a^2 + b^2 = c^2$. I have assumed $b \geqslant a$ to avoid repetitions; also if $a = b$, then $c^2 = 2a^2$ which contradicts the condition that a and c are integers. Thus the condition $a < b < c$ can be imposed without loss of generality.

I will define the order of the triple (a,b,c) to be n if $c = b + n$. For example the triple (3,4,5) is a first order triple, and in fact the first order triples are given in the solution to problem 261 (see Vol. 11 No. 2). In this article, I will show how one can systematically find all triples of order 1, 2, . . . , n.

If (a,b,c) is a triple of order n, then $c = b + n$ and so

$$a^2 = (b+n)^2 - b^2 = 2bn + n^2 = (2b+n)n \qquad (1)$$

Since $a < b$, $a^2 - n^2 = 2bn > 2na$ and so

$$a^2 - 2na - n^2 > 0.$$

Since this quadratic has roots $a = n(1 \pm \sqrt{2})$ and $a > 0$, we see that $a > n(1 + \sqrt{2})$. This gives a lower bound for a.

Also, from equation (1), we can see that n divides a^2. Now it can be shown that if p is any prime which divides n, then p divides a^2 and so p divides a. Thus, in the case where $n = p_1 p_2 \ldots p_r$ where $p_1, p_2, \ldots, p_r$ are different prime numbers, then n divides a and, if n is odd, n also divides $b = (a^2 - n^2)/2n$ and so n divides b and c. If n is odd, $2b + n$ is odd and so $a^2 = (2b + n)n$ is odd. Again it can be shown that, since a^2 is odd, a is odd. If n is not of the form $p_1 p_2 \ldots p_r$, we need to examine the factors of n, e.g. if $n = 360 = 2^3 \times 3^2 \times 5$, we can see that a is divisible by $2^2 \times 3 \times 5 = 60$ and $a > 360(1 + \sqrt{2}) > 864$.

Some examples of Pythagorean triples may now be determined. For example, the first order triples are given by $n = 1$, $b = (a^2 - 1)/2$, a is odd and $a > 1 + \sqrt{2}$: i.e. (3,4,5), (5,12,13), (7,24,25), (9,40,41), . . . Similarly the second order triples are given by $n = 2$, $b = (a^2 - 4)/4$, a is even and $a > 2(1 + \sqrt{2}) > 4.8$; i.e. (6,8,10), (8,15,17), (10,24,26), . . .

We can also answer some of Mr Ryan's questions:

(1) To find the Pythagorean triples whose smallest number is 15, we let $a = 15 = 3 \times 5$. As n divides a^2, the possible values for n are 3, 9, 5, 25, 15, 45, 75 and 225. But $n(1 + \sqrt{2}) < a = 15$ and so $n < 15/(1 + \sqrt{2}) < 7$. Thus $n = 3$ or 5 and the triples are (15,36,39) and (15,20,25).

(2) Similarly, if $a = 45$ then $n < 19$ and n divides 2025. The only triples are (45,336,339), (45,108,117), (45,200,205) and (45,60,75).

T. Kelley
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Some Match-stick Problems
(Contributed by Diana Eilert of Woden Valley High)

(i) Four cows and three tigers are in the same stockade. With three matches separate the animals so that each is in a pen of its own (the tigers have stripes!)

(ii) A real estate agent wanted to sub-divide his block of L shaped land into four equal L shaped portions, but was unable to think of a way to do it. He offered a $100 prize for a solution, and to his surprise and delight his son won the prize. Representing the piece of land by 16 matches, try to divide it into 4 equal L shaped portions using another eight matches.

(iii) Make an equilateral triangle with nine matches. See if you can make it five-ninths its original size by moving only three matches.

(Answers on page 28)