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FALLACIOUS PROOFS AND PARADOXES

A chain is as strong as its weakest link, and this applies particularly to the chain of a mathematical argument. Sometimes weaknesses are hidden behind the subtleties of an argument, and if the conclusion of the argument is not grossly absurd these weaknesses may remain undetected for years or even decades. However, if the argument leads to an absurd or paradoxical conclusion, every definition, assumption, or logical step which forms a link in the chain of argument must be carefully scrutinised. Whole chapters of mathematics have been opened up as the result of such scrutiny.

To illustrate how an argument can go wrong, we give some examples, some of which may be known to you. In each case you can see that there must be an error somewhere, because the conclusion is nonsensical. See if you can find the errors yourselves, and then compare your findings with the answers given elsewhere in this issue.

Example 1: Three students had lunch at a restaurant and, as the bill came to \$3, each student gave the waiter \$1. However, when the waiter went to the office, he discovered that the bill only came to \$2.50. As he could not be bothered dividing 50c three ways, the waiter kept 20c of the change and gave the students 10c change each. Thus, the students paid 3 x 90c = \$2.70 and the waiter kept 20 making \$2.90 in all. What happened to the other 10c?

Example 2: Any number, a, is equal to its double, 2a.

Proof: Let $x = a$

Multiplying by a, $ax = a^2$

Subtracting x^2 from both sides, $ax - x^2 = a^2 - x^2$

that is, $x(a-x) = (a+x)(a-x)$.

Cancelling, we have $x = a + x$,

and since $x = a$ this means that

$$a = a + a = 2a.$$

Example 3: By multiplying out, we can verify that for all n

$$n^2 - n(2n+1) = (n+1)^2 - (n+1)(2n+1).$$

We complete the square by adding $[\frac{1}{2}(2n+1)]^2$ to both sides:

$$n^2 \quad n(2n+1) + [\tfrac{1}{2}(2n+1)]^2 = (n+1)^2 - (n+1)(2n+1) + [\tfrac{1}{2}(2n+1]^2$$

that is $$[n - \tfrac{1}{2}(2n+1)]^2 = [n + 1 - \tfrac{1}{2}(2n+1)]^2$$

Hence, taking square roots,

$$n-\tfrac{1}{2}(2n+1) = n+1-\tfrac{1}{2}(2n+1).$$

and so for all n

$$n = n+1,$$

i.e. a number is unchanged when 1 is added to it.

Example 4: We have $2 > 1$ and $\log \tfrac{1}{2} = \log \tfrac{1}{2}$

Hence $2 \log \tfrac{1}{2} > 1 \log \tfrac{1}{2}$

that is, $\log (\tfrac{1}{2})^2 > \log (\tfrac{1}{2})^1$

Hence $(\tfrac{1}{2})^2 > (\tfrac{1}{2})^1$

that is, $\tfrac{1}{4} > \tfrac{1}{2}$.

Example 5: Every triangle is isosceles.

Proof: Let the triangle be ABC and let the bisector of the angle A and the perpendicular bisector of the side BC intersect at M. There are two possible cases:

(a) M is inside the triangle (Figure 1);

(b) M is outside the triangle (Figure 2).

(We may ignore the case when M is on BC, since it is then easy to show legitimately that the triangle is isosceles.) Let O be the midpoint of BC, and X, Y be the feet of the perpendiculars from M on AB, AC, respectively.

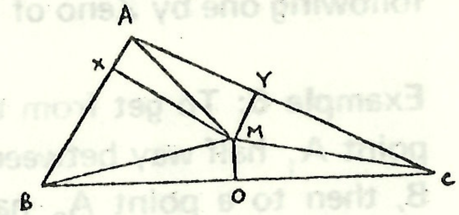


Figure 1

Since MO is the perpendicular bisector of BC,

$$\Delta OMB \equiv \Delta OMC \quad \text{(side-angle-side)}$$

and hence

$$MB = MC.$$

Since MA bisects $\angle A$ and the angles at X, Y are right angles and MA is common,

$$\Delta MAX = \Delta MAY.$$

Hence

$$MX = MY,$$

and

$$AX = AY. \quad (1)$$

Since the angles at X, Y are right angles and we have shown MB = MC and MX = MY, we have

$$\Delta MBX \equiv \Delta MCY,$$

and so

$$BX = CY \quad (2)$$

Adding equations (1) and (2)

$$AX + BX = AY + CY,$$

that is,

$$AB = AC.$$

(b) If M is outside the triangle, the above proof can still be repeated in every detail, except that the equations (1) and (2) must now be subtracted:

$$AX - BX = AY - CY,$$

hence

$$AB = AC$$

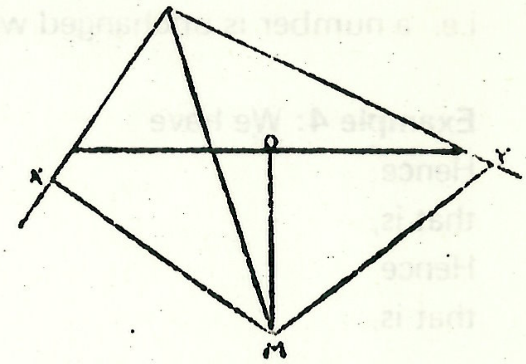


Figure 2

Some paradoxes have led to a deeper understanding of mathematics such as the one mentioned in the letter from John Rogers (see "Your Letters") and the following one by Zeno of Elea in the fifth century B.C.:

Example 6: To get from the point A to the point B, we must first go from A to a point A_1 half way between A and B, then to a point A_2 half way between A_1 and B, then to a point A_3 half way between A_2 and B, and so on. Since this is an infinite process, we can never get from A to B. Thus motion is impossible.

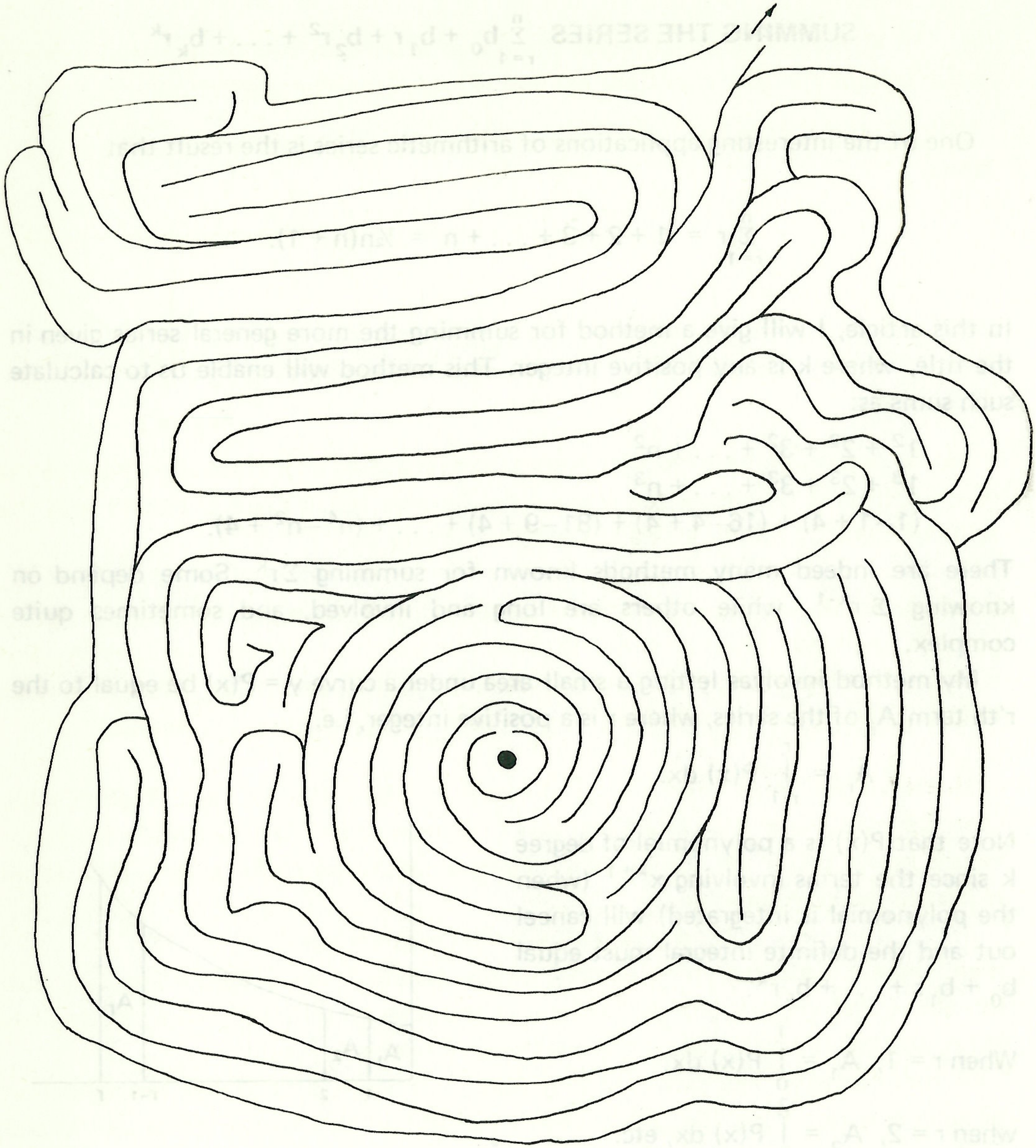
Some of the most difficult paradoxes are to be found in logic. Probably the most famous one is the statement "All Cretans are liars" by the Cretan (and who would know better than a Cretan?) Epimenides in the sixth century B.C. If we re-phrase this statement as

"Every statement by a Cretan is false"

then, since it was made by a Cretan, it must be false. So we deduce logically that the above statement is false (whether it is or not in actual fact!). This paradox has led to the law of logic that a statement may not refer to itself.

A more modern example is that of Burali-Forti who proved that the collection of all sets is not a set. If it were, then we could write S for the set of all sets and P(S) for the set of all subsets of S (e.g. $S \in P(S)$). Since S is the set of all sets and P(S) is a set, $P(S) \in S$, which can be shown to be impossible. This and similar paradoxes have led to a great deal of debate amongst Mathematicians which is still going on today.

Some books: "Fallacies in Mathematics" by E.A. Maxwell (Cambridge); "Riddles in Mathematics" by E.P. Northrop (Penguin).



Find your way out of the maze starting at the dot. (Answer next issue)