SUMMING THE SERIES $\sum_{r=1}^{n} b_0 + b_1 r + b_2 r^2 + \ldots + b_k r^k$

One of the interesting applications of arithmetic series is the result that

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1).$$

In this article, I will give a method for summing the more general series given in the title, where k is any positive integer. This method will enable us to calculate such sums as:

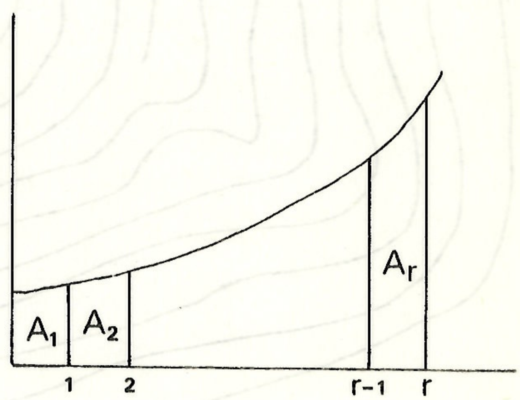
$$1^2 + 2^2 + 3^2 + \ldots + n^2$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3$$
$$(1-1+4) + (16-4+4) + (81-9+4) + \ldots + (n^4 - n^2 + 4).$$

There are indeed many methods known for summing Σr^k. Some depend on knowing $\Sigma\, r^{k-1}$, while others are long and involved, and sometimes quite complex.

My method involves letting a small area under a curve $y = P(x)$ be equal to the r'th term A_r of the series, where r is a positive integer, i.e.

$$A_r = \int_{r-1}^{r} P(x)\, dx.$$

Note that $P(x)$ is a polynomial of degree k since the terms involving x^{k+1} (when the polynomial is integrated) will cancel out and the definite integral must equal $b_0 + b_1 r + \ldots + b_k r^k$.



When $r = 1$, $A_1 = \int_0^1 P(x)\, dx$;

when $r = 2$, $A_2 = \int_1^2 P(x)\, dx$, etc.

Adding all these areas we get:

$$\sum_{r=1}^{n} b_0 + b_1 r + \ldots + b_k r^k = A_1 + A_2 + \ldots + A_n$$

$$= \int_0^1 P(x)\,dx + \int_1^2 P(x)\,dx + \ldots + \int_{n-1}^{n} P(x)\,dx$$

$$= \int_0^n P(x)\,dx.$$

If $P(x) = a_0 + a_1x + \ldots + a_kx^k$, then

$$A_r = \int_{r-1}^{r} P(x)\,dx = \int_{r-1}^{r} (a_0 + a_1x + \ldots + a_kx^k)\,dx$$

$$= [\,a_0x + a_1x^2/2 + \ldots + a_kx^{k+1}/(k+1)\,]_{r-1}^{r}$$

$$= b_0 + b_1r + \ldots + b_kr^k.$$

Thus only the simplest form of the binomial theorem (powers of $r-1$) is needed. Once the co-efficients $a_0, a_1, \ldots a_k$ are known, it is just a matter of substituting n in the integral of P(x).

A simple example is the series $1^2 + 2^2 + \ldots + n^2$. If

$$r^2 = \int_{r-1}^{r} (a_0 + a_1x + a_2x^2)\,dx = a_0 + a_1(2r-1)/2 + a_2(3r^2-3r+1)/3,$$

then (equating co-efficients) $a_2 = 1$, $a_1 = 1$, $a_0 = 1/6$.

$$\text{Thus } 1^2 + 2^2 + 3^2 + \ldots + n^2 = \int_0^n (1/6 + x + x^2)\,dx = (n + 3n^2 + 2n^3)/6.$$

This method may be generalised by using for A_r the area under the curve $y = P(x)$ between $r-m$ and $r + l$, where $l + m = 1$, instead of between $r-1$ and r. In this case, the sum will be $\int_l^{n+l} P(x)\,dx$. A value for l must now be found to make calculations as simple as possible. [$l = ½$ or $l = 1$ sometimes makes the calculations easier than $l = 0$ as Greg has chosen – Editor.]

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