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YOUR LETTERS

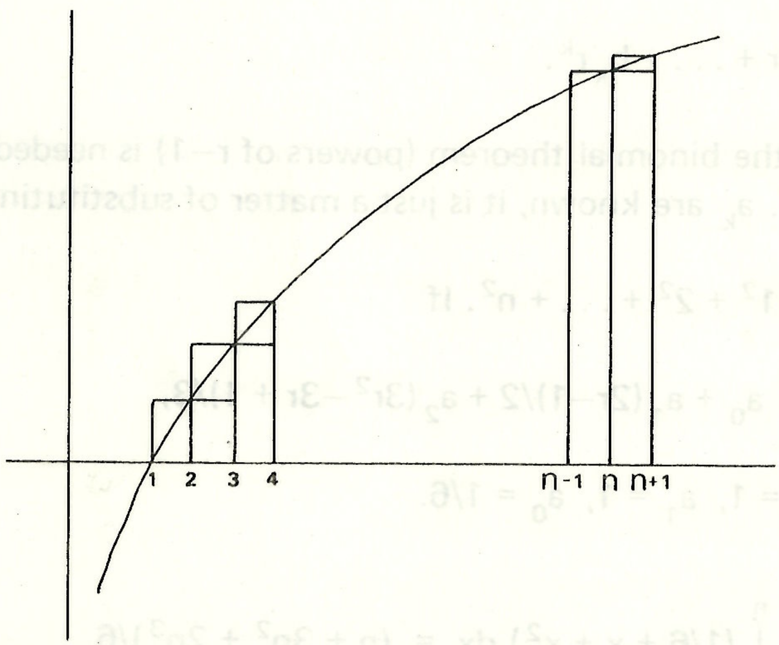
Dear Sir,

Recently I found the following relationships between n! and n^n while working through sequences and series:

$$\text{First,}\quad 0 \leqslant n!/n^n = \frac{1 \,.\, 2 \,.\, 3 \ldots (n-1) \,.\, n}{n \,.\, n \,.\, n \ldots \; n \;\; . \, n}$$

$$< 1/n \quad \text{for } n > 2.$$

$$\text{Thus } \lim_{n\to\infty} n!/n^n = \lim_{n\to\infty} 1/n = 0.$$



Now, by considering the area under the curve y = log x and the areas of the rectangles shown in the figure, we can see that

log n!

$= \log 2 + \log 3 + \ldots + \log n$

= sum of rectangles between 2 and n + 1 below curve

$$< \int_1^{n+1} \log x \, dx$$

$$= (n+1)\log(n+1) - n$$

Similarly, log n! = sum of rectangles between 1 and n above curve

$$> \int_1^n \log x \, dx = n \log n - n + 1$$

$$\text{Thus } 1 - 1/(\log n) + 1/(n \log n) < (\log n!)/(\log n^n)$$

$$< (1 + 1/n) \log (n+1)/(\log n) - 1/(\log n)$$

as $n \to \infty$, the left hand and right hand expressions tend to 1, and so

$$\lim_{n\to\infty} (\log n!)/(\log n^n) = 1.$$

$$\text{Thus} \quad \lim_{n\to\infty} \log n! = \lim_{n\to\infty} \log n^n$$

$$\lim_{n\to\infty} n! = \lim_{n\to\infty} n^n$$

and so $$\lim_{n\to\infty} n!/n^n = 1.$$

This is a different answer to the previous one. How do you explain this contradiction?

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[The explanation is given on page 27 – Editor.]

Dear Sir,

I have discovered an interesting theorem which appears to have several applications. It is:

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

where ^nC_r means the number of ways of selecting r articles from n articles.

To prove the theorem, you use Pascal's triangle:

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         1           =  1
       1 + 1         =  2
     1 + 2 + 1       =  4
   1 + 3 + 3 + 1     =  8
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Each element in the triangle is the sum of the two elements diagonally above it.

i.e. $^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

Thus each element in the (n-1)st row contributes twice to the sum of the elements in the n'th row and so (by using induction) we can see that

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = \text{sum of elements in n'th row} = 2^n$$

One application of this result is a means of proving that a set of n elements has 2^n subsets (see question 3 of the Senior Division of the Mathematics Competition 1975). No. of subsets = no. of subsets with 0 elements + no. of subsets with 1 element + . . . + no. of subsets with n elements = $^nC_0 + {}^nC_1 + \ldots + {}^nC_n = 2^n$.

Another application is to a matrix A with $A^2 = A$. Since $A^2 = A$, we can see that $A = A^2 = A^3 = \ldots = A^n$. Thus

$$(I + A)^n = {}^nC_0 I + ({}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)A = I + (2^n - 1)A.$$

Does anyone else know other problems in which this is useful?

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[An easier proof is to expand $2^n = (1 + 1)^n$ by the Binomial Theorem – Editor.]