MAGIC SQUARES

One of the most interesting, and oldest, recreations in Mathematics is the magic square. It consists of a number of different positive integers arranged in the form of a square so that the sum of the integers in any row, column or diagonal is the same. The example in Figure 1 was discovered by the Chinese Emperor Yu about 2000 B.C. who called it the lo-shu, and the example in Figure 2 was used by the sixteenth century artist Albrecht Dürer in a picture called "Melencolia". The probable reason for this is that astrologers of that period associated magic squares with Saturn which they believed to be related to melancholy. See if you can guess what year Dürer engraved his picture. (Hint: Look at the middle numbers in the bottom row.)

8	1	6
3	5	7
4	9	2

Figure 1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Figure 3

The example in Figure 3 was found in an inscription at Khajuraho, India and is the earliest recorded magic square using the numbers 1, 2, . . . 16 (it was inscribed about 1200 A.D.).

About 40 years ago, the U.S. architect C. Bragdon found that magic squares produce a fascinating pattern by joining the numbers of the square in the correct order and you might like to try this with the magic squares given in this article.

In Figure 1 the sum of all the numbers in any row or any column or the two diagonals (8+5+2 and 4+5+6) is 15. Similarly the numbers in any row, any column or either diagonal of Figure 2 or Figure 3 add up to 34. In fact, if we arrange the numbers from 1 to n^2 in a magic square (called a magic square of order n) then, by the method used in problem 285, we can see that all the numbers must add up to $\frac{1}{2}n^2(n^2+1)$ and so the sum of each row, column or diagonal must be $\frac{1}{2}n(n^2+1)$. Of course, the set of integers we use does not have to be $\{1,2, \ldots n^2\}$ but this will do for a start.

How many magic squares are there? It is quite easy to see that it is impossible to make one of order 2, and examples of order 3 and 4 are given in Figures 1, 2, 3. In fact, if we try to make another lo-shu (or magic square of order 3) it is not hard to see that each of the 18 possibilities is either a rotation of the one given or a reflection of it in one of its axes. Similarly, it has been shown that there are 7040 magic squares of order 4 but that they fall into 880 sets of 32 "isomorphic" magic squares (i.e. magic squares which are the same except for a rotation or a reflection). No one knows how many magic squares there are of order 5, 6, 7, . . . At last count, there were over 13 million of order five!

Let us see if we can construct some magic squares. A way to make one of order 5 is to write the number 1 in the middle of the top row, 2 in the bottom of the next column and so on, continuing along a diagonal line sloping upwards and to the right. When we reach an edge, we proceed as though the square had been drawn again at that edge (see Figure 4a) and when we reach a position already occupied by a number we drop down one row and start again (see Figure 4b).

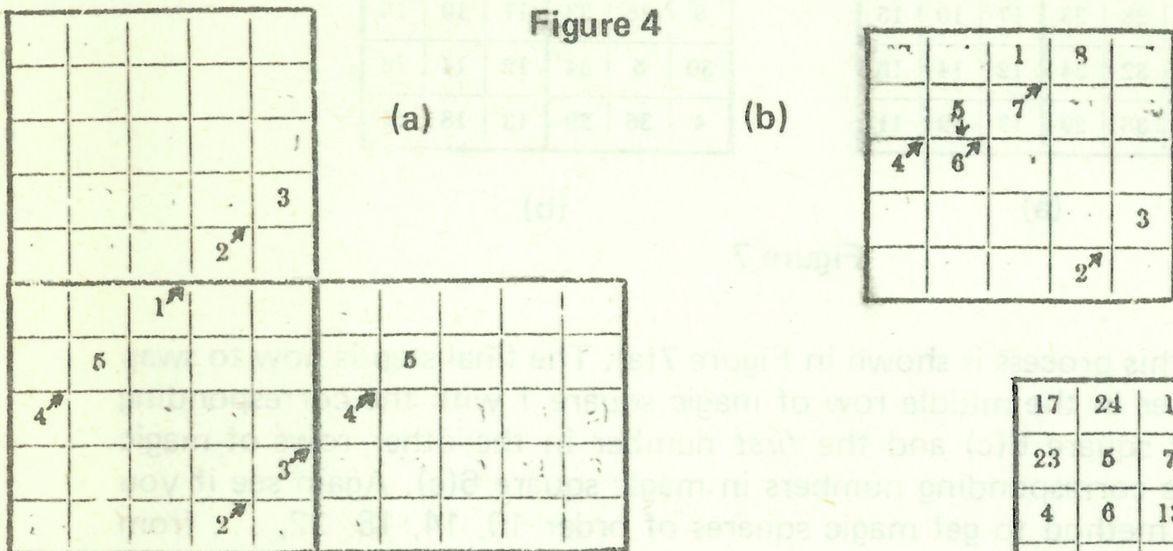


Figure 4

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 5

The completed magic square is shown in Figure 5. You might like to try the same method to construct a magic square of order 7 (it works for magic squares of any odd order).

Making a magic square of order 6 is a little more tricky. First note that by adding 9, 18 and 27 to all the numbers in the magic square of Figure 1, we get the following magic squares (whose sums are 15+36, 15+72 and 15+108 respectively):

(a)

17	10	15
12	14	16
13	18	11

(b)

26	19	24
21	23	25
22	27	20

(c)

35	28	33
30	32	34
31	36	29

Figure 6

Note that all the integers from 1 to 36 are included in Figures 1, 6(a), 6(b) and 6(c). The second step is to assemble the four magic squares using the scheme:

1	6 (b)
6 (c)	6 (a)

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	10	15
30	32	34	12	14	16
31	36	29	13	18	11

(a)

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

(b)

Figure 7

The result of this process is shown in Figure 7(a). The final step is now to swap the *second* number in the middle row of magic square 1 with the corresponding number in magic square 6(c) and the *first* number in the other rows of magic square 1 with the corresponding numbers in magic square 6(c). Again see if you can modify this method to get magic squares of order 10, 14, 18, 22, . . . from those of order 5, 7, 9, 11, . . .

So far, we have seen a method for constructing a magic square whose order is an odd integer or twice an odd integer. The remaining magic squares have orders divisible by 4 and so we now look at that case. The smallest one has order 4 and we can construct one very similar to Durer's by the following method: Write down the integers from 1 to 16 in order on the square and swap each number on the diagonals with the number "symmetrically opposite" it (see Figure 8). Thus 1 and 16, 4 and 13, 6 and 11, 7 and 10 are swapped.

	2	3	
5			8
9			12
	14	15	

→

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figure 8

To use this method on a magic square of order 8, we write all the integers from 1 to 64 in order on the square as before. However, in this case we first break the large square up into four smaller squares of 16 numbers as in the last case and change the diagonal numbers *of each smaller square* by replacing the number n by the number 65–n (see Figure 9).

	2	3			6	7	
9			12	13			16
17			20	21			24
	26	27			30	31	
	34	35			38	39	
41			44	45			48
49			52	53			56
	58	59			62	63	

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

Figure 9

Some Special Magic Squares

If we look at the examples we have so far constructed, we will see that some of them possess more features.

1. **Symmetrical Squares.** If we look at Durer's magic square, we notice that any pair of symmetrically opposite numbers add up to the same number. Thus 16 + 1 = 3 + 14 = 2 + 15 = 13 + 4 = 10 + 7 etc. Such a magic square is called symmetrical and, because of the methods we have used, all the magic squares in Figures 1, 2, 5 and 8 are symmetrical (the symmetrically opposite numbers add up to 10, 17, 26 and 17 respectively). The reason the magic square of order 6 is not symmetrical is because no magic square whose order is twice an odd number can be symmetrical. You may care to find how many symmetrical magic squares there are of order 4 or 5.

2. **Pandiagonal Squares.** If we draw five copies of Figure 3 in the form of a cross, we notice that any 4 consecutive numbers in any diagonal line again add up to 34. For example see Figure 10.

Such magic squares are called pandiagonal ("pan" is Greek for "all"). None of the other examples given above are pandiagonal, but you might like to see if you can construct some more (Figure 10 will give you another 15 examples if you look carefully). There are no pandiagonal magic squares of order 6, 10, 14, . . . or of order 3, but there are 48 of order 4 and 3600 of order 5 (not counting rotations and reflections).

$$9 + 12 + 8 + 5 = 6 + 1 + 11 + 16 = 15 + 12 + 2 + 5 = 34$$

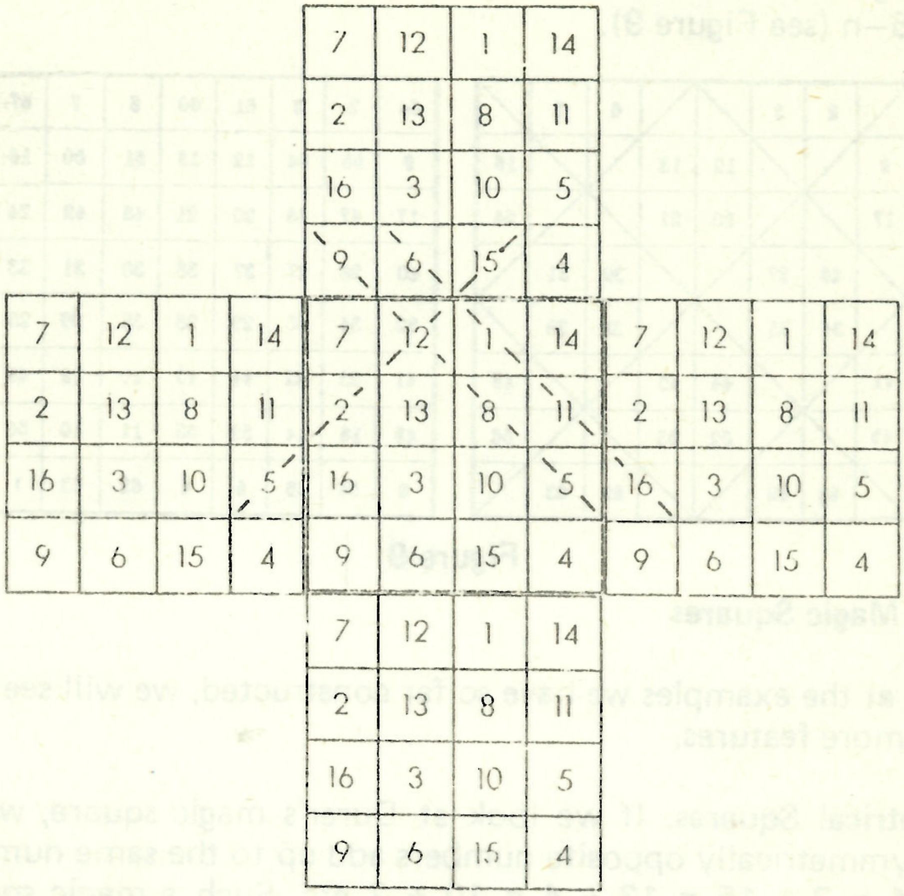


Figure 10

3. **Doubly Magic Squares.** You might like to take up the challenge of constructing a magic square which remains a magic square when each number is replaced by the square of that number. Such magic squares are called doubly magic and the smallest possible one has order 8. Thus the lo-shu (Figure 1) is not doubly magic since $8^2 + 1^2 + 6^2 = 101 \neq 83 = 3^2 + 5^2 + 7^2$.

4. A fascinating new type of magic square is described by Martin Gardner in Scientific American (January, 1957) and is shown in Figure 11.

Choose any number, circle it and cross out all the numbers in that row and that column. Choose any number which has not been crossed out and repeat the above process. Continue until all numbers have been crossed out and add up the 5 circled numbers: the answer will be 62 (the sum of any row or column). For example: 9 + 17 + 5 + 28 + 3 = 62.

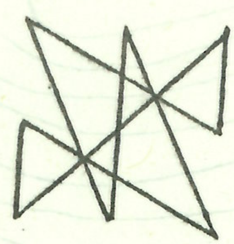
20	9	12	26	8
13	2	5	19	1
17	6	9	23	5
22	11	14	28	10
15	4	7	21	3

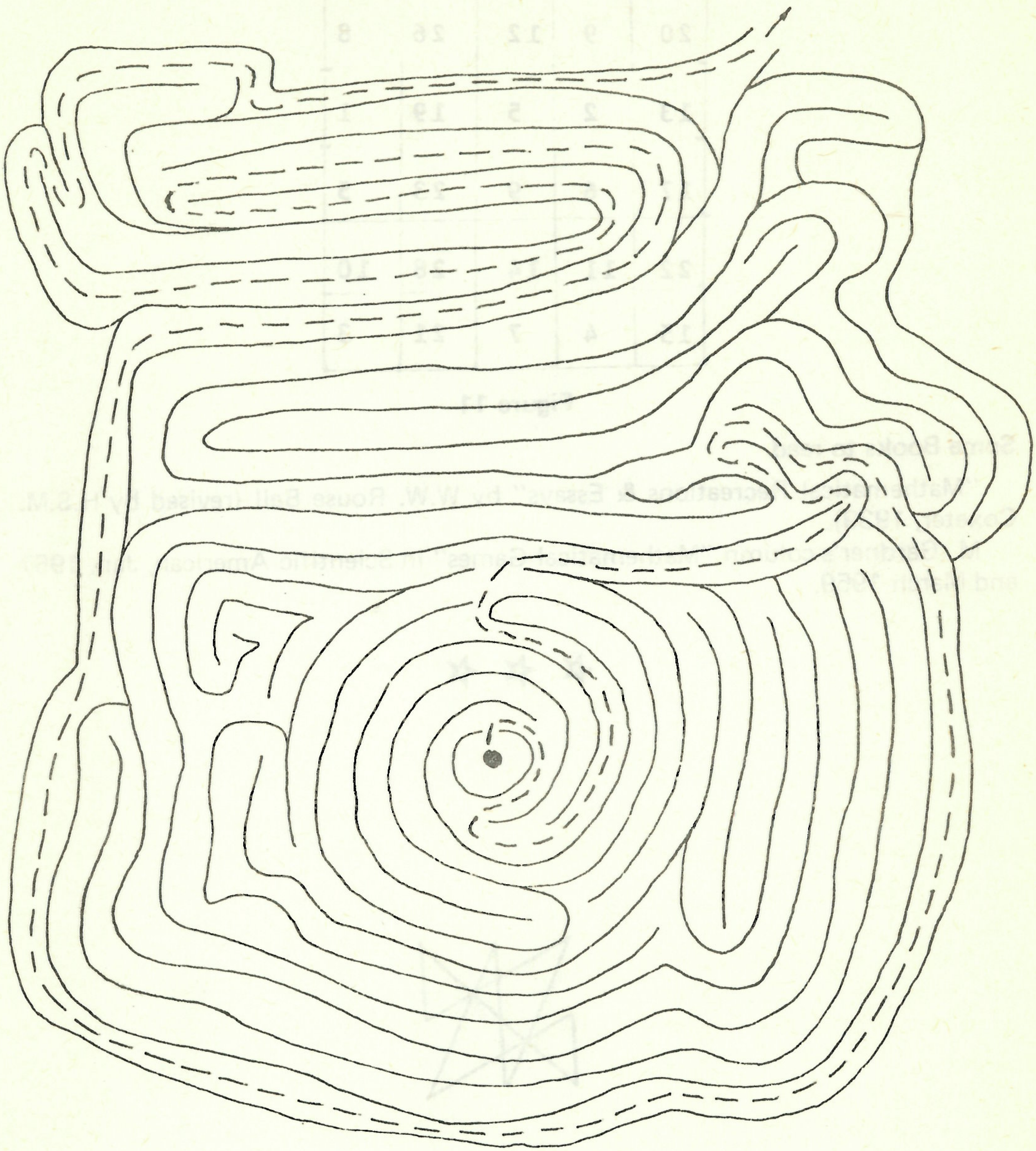
Figure 11

Some Books to read

"Mathematical Recreations & Essays" by W.W. Rouse Ball (revised by H.S.M. Coxeter, 1939).

M. Gardner's column "Mathematical Games" in Scientific American, Jan. 1957 and March 1959.





Solution to maze in Vol. 11 No. 3