FIBONACCI NUMBERS, PASCAL'S TRIANGLE, AND PRIME NUMBERS

The famous Fibonacci numbers are a sequence of numbers defined by $T_1 = 1$, $T_2 = 1$, and $T_n = T_{n-1} + T_{n-2}$ for $n = 3, 4, 5, \ldots$. That is to say, each term after the second is the sum of the previous two terms. Thus, the sequence continues $T_3 = 2$, $T_4 = 3$, $T_5 = 5$, $T_6 = 8$, $T_7 = 13$, and so on. The Fibonacci sequence, you will observe, is neither an arithmetic nor a geometric progression. Indeed, it was one of the first such sequences studied.

It is natural to ask whether there is a formula for T_n, by which it may be possible to calculate T_n for some large values of n without having to work out all the previous terms. The answer is yes: indeed, I shall give two, apparently different, formulas for T_n.

The first formula is

$$T_n = (a^n - b^n)/(a-b) \text{ where } a = \tfrac{1}{2}(1+\sqrt{5}),\ b = \tfrac{1}{2}(1-\sqrt{5}) \qquad (1)$$

Having been told this, you may now prove it by induction (see exercise 1). But the question is, how did I find this formula? And that is what I will start by showing you. It is often useful when studying a series like the Fibonacci series to write down its so-called "generating function", viz.

$$f(x) = T_1 + T_2x + T_3x^2 + T_4x^3 + \ldots \qquad (2)$$

where the co-efficients are the Fibonacci numbers.

[NOTE: This is an infinite series and we should worry about its convergence. However, since we are not going to substitute any values for x, we can ignore this problem. In fact, it can be shown that if $-\frac{1}{2} < x < \frac{1}{2}$ then this series actually does converge. – Ed.]

Since the co-efficients are the Fibonacci numbers,

$$f(x) = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + \ldots$$

Therefore

$$x\,f(x) = 1x + 1x^2 + 2x^3 + 3x^4 + \ldots$$

and

$$x^2 f(x) = 1x^2 + 1x^3 + 2x^4 + \ldots$$

Now observe that, surprisingly, $f(x) - xf(x) - x^2 f(x) = 1$, since all the other terms on the right hand side cancel out!

So $\qquad (1-x-x^2)\,f(x) = 1,$

or $\qquad f(x) = 1/(1-x-x^2).$ $\qquad$ (3)

All the Fibonacci numbers are bottled up in this neat little expression! All we have to do to release them is a division:

$$
\begin{array}{r|l}
 & 1 + 1x + 2x^2 + \ldots \\
\hline
1-x-x^2 & 1 \\
 & 1 - x - x^2 \\
\hline
 & \quad x + x^2 \\
 & \quad x - x^2 - x^3 \\
\hline
 & \qquad 2x^2 + x^3 \\
 & \qquad 2x^2 - 2x^3 - 2x^4 \\
\hline
 & \qquad\quad 3x^3 + 2x^4 \\
 & \qquad\qquad \vdots
\end{array}
$$

We must be able to do something with the fact that

$$f(x) = 1/1(1-x-x^2).$$

Let us write $1-x-x^2 = (1-ax)(1-bx)$, a rather unusual sort of factorisation. We have $a + b = 1$, and $ab = -1$, and so a, b are the roots of $z^2-z-1 = 0$, i.e. $z = \tfrac{1}{2}(1+\sqrt{5})$ or $\tfrac{1}{2}(1-\sqrt{5})$. Let us set $a = \tfrac{1}{2}(1+\sqrt{5})$ and $b = \tfrac{1}{2}(1-\sqrt{5})$. So we have $f(x) = 1/(1-ax)(1-bx) = 1/(1-ax) \times 1/(1-bx)$. Now, observe that

$$
\begin{aligned}
(1-ax)(1+ax+a^2x^2+a^3x^3+\ldots) &= 1 + ax + a^2x^2 + a^3x^3 + \ldots \\
&\quad - ax - a^2x^2 - a^3x^3 - \ldots \\
&= 1
\end{aligned}
$$

Thus $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \ldots$, and similarly $1/(1-bx) = 1 + bx + b^2x^2 + b^3x^3 + \ldots$ So

$$
\begin{aligned}
f(x) &= (1 + ax + a^2x^2 + a^3x^3 + \ldots)(1 + bx + b^2x^2 + b^3x^3 + \ldots) \\
&= 1 + (a+b)x + (a^2 + ab + b^2)x^2 + (a^3 + a^2b + ab^2 + b^3)x^3 + \ldots
\end{aligned}
$$

Comparing this with equation (2), we see that

$$
\begin{aligned}
T_n &= a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1} \\
&= (a^n - b^n)/(a-b)
\end{aligned}
$$

So formula (1) is proved!

Pascal's Triangle and Fibonacci Numbers

Another formula for T_n can be found from Pascal's triangle. Suppose we displace the rows of Pascal's triangle and add the columns as follows:

1									
	1	1							
		1	2	1					
			1	3	3	1			
				1	4	6	4	1	
					1	5	10	10	5
						1	. . .		
1	1	2	3	5	8	13	. . .		

The column sums are just the Fibonacci numbers!

This gives the following formula for T_n:

$$T_n = 1 + {}^{n-1}C_1 + {}^{n-2}C_2 + \dots + {}^{\frac{1}{2}n}C_{\frac{1}{2}n} \qquad \text{if n is even,}$$

$$T_n = 1 + {}^{n-1}C_1 + {}^{n-2}C_2 + \dots + {}^{\frac{1}{2}(n+1)}C_{\frac{1}{2}(n-1)} \qquad \text{if n is odd}$$

(where of course ${}^nC_r$, or $\binom{n}{r}$ denotes the binomial coefficient, as usual).

We can write these two formulas neatly as one:

$$T_n = \sum_{r \leqslant \frac{1}{2}n} {}^{n-r}C_r. \tag{4}$$

This formula is proved as follows:

If we write y for $x + x^2$, equation (3) becomes

$$\begin{aligned} f(x) &= 1/(1-y) \\ &= 1 + y + y^2 + y^3 + \dots \\ &= 1 + (x + x^2) + (x + x^2)^2 + (x + x^2)^3 \dots \\ &= 1 + x(1 + x) + x^2(1 + x)^2 + x^3(1 + x)^3 + \dots \\ &= 1 \\ &\quad + x + x^2 \\ &\quad\quad + x^2 + 2x^3 + x^4 \\ &\quad\quad\quad + x^3 + 3x^4 + 3x^5 + x^6 \\ &\quad\quad\quad\quad + x^4 + \dots \end{aligned}$$

and if you look closely you will see Pascal's triangle displaced, giving the formula (4), since $f(x) = T_1 + T_2 x + T_3 x^2 \ldots$

Prime Numbers and Pascal's Triangle

Now I want to show you a connection between Pascal's triangle and prime numbers which was discovered only recently.

Suppose we displace the rows of Pascal's triangle yet further, as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1														
1			1	1											
2					1	2	1								
3							1	3	3	1					
4									1	4	6	4	1		
5											1	5	10	10	5
6													1	6	15
7															1

To see if a number n is prime or not, look at the numbers in the column below n. Thus, under 11, we have 4 and 5. Check each of these numbers in turn to see if they are divisible by their row number m. Thus, under 11, the 4 is divisible by its row number (4), and the 5 is divisible by its row number (5). If, as, in the case of 11, every number in column n is divisible by its row number, then the number n is prime. If, on the other hand, any one of the numbers in column n is not divisible by its row number, then n is not prime.

The numbers in column 17 are 6 (in row 6), 35 (in row 7) and 8 (in row 8) and so 17 is prime; while in column 25, the numbers are 36 (in row 9), 252 (in row 10), 165 (in row 11) and 12 (in row 12), and since 252 is not divisible by 10, 25 is not prime.

Prime Numbers and Fibonacci Numbers

Finally I would like to show you a connection between the Fibonacci numbers and the primes: if p is a prime which ends in the digit 1 or 9, then T_p is one more than a multiple of p, while if p is a prime which ends in a 3 or a 7, then T_p is one less than a multiple of p. For example,

$$T_{11} = 89 = 8 \times 11 + 1$$
$$T_{13} = 233 = 18 \times 13 - 1$$
$$T_{17} = 1597 = 94 \times 17 - 1$$
$$T_{19} = 4181 = 220 \times 19 + 1.$$

Exercises

1. Let P(n) be the proposition "$T_n = (a^n - b^n)/(a-b)$ *and* $T_{n+1} = (a^{n+1} - b^{n+1})/(a-b)$", where a, b are the numbers in formula (1). Use induction to show that P(n) is true for all $n \geqslant 1$.
2. Show that $T_{n+1}/T_n \to \frac{1}{2}(1 + \sqrt{5})$ as $n \to \infty$.
3. Find a formula for S_n, the sum of the first n Fibonacci numbers. Show that $S_{n+1}/S_n \to \frac{1}{2}(1 + \sqrt{5})$ as $n \to \infty$.
4. Use Pascal's triangle to find the first 20 Fibonacci numbers.
5. Apart from the case $T_4 = 3$, it *is* true that if T_n is prime, then n is prime. Is it true that apart from the case n = 2, if n is prime, T_n is prime?
6. Use Pascal's triangle to find the first 20 primes.
7. A sequence U_n is defined by $U_1 = 1$, $U_2 = 2$, $U_n = 2U_{n-1} - U_{n-2}$ for n = 3, 4, 5, ... Find a formula for U_n using the generating function. Prove your formula by induction.

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