PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions, and the names of those who submit solutions before the publication of the next issue, will be published in Vol. 12 No. 2.

Although the problems are unclassified, the first 4 problems should require no more mathematical knowledge than is learnt in first or second form and the next 4 problems should require no more mathematical knowledge than is learnt in the first 4 forms of High School. Senior students are thus encouraged to attempt the later problems.

297. A man has 3 bottles which hold exactly 8 litres, 5 litres and 3 litres. The two smaller bottles are empty but the largest one is full of wine which the man wishes to share with a friend. Without using any other means of measurement or any other container, how can he divide the wine into two equal amounts of 4 litres each?

298. $1^3 + 2^3 + 3^3 = 36$, which is divisible by 18. Find all sets of three consecutive natural numbers such that the sum of their cubes is divisible by 18.

299. It is not hard to show that the trinomial $x^4 + px^2 + q$ is divisible by $x^2 + 1$ if and only if $p = q + 1$. For the general quadratic $x^2 + ax + b$, find the values of p, q such that $x^4 + px^2 + q$ is divisible by $x^2 + ax + b$.

300. Prove that if $1/(ab) + 1/(bc) + 1/(ca) = 1/(ab + bc + ca)$ then the sum of two of the numbers a, b and c is zero.

301. By moving the digits in the following magic square (and using no other operation), find 9 numbers in 3 rows and 3 columns such that when the numbers in any row or column or diagonal are *multiplied* together, you get the same answer. For example, the top row might be replaced by 7, 202 and 52. Magic squares are discussed in the article on page 2.

27	20	25
22	24	26
23	28	21

302. A, B, C and D, are four points, in that order, on a straight line.

(i) If AB* = CD*, show that for any point P in the plane, PA* + PD* ≥ PB* + PC*.

(ii) Conversely, if PA* + PD* ≥ PB* + PC* for every position of P, show that AB* = CD*.

303. Four men A, B, C and D set out simultaneously from M to reach N, 5 kilometres away. One of them, D, owns a motorcycle. He gives A a lift for part of the way, then turns back and picks up B. When they overtake A, B alights and the unselfish D once more turns back to assist C. Eventually they all arrive at N at the same moment. If D always travels at a steady v km/hour, and A, B, and C all walk at w km/hr, how long did the trip from M to N take?

304. Prove that a ray of light, having been reflected from three mutually perpendicular mirrors in turn becomes parallel to its original direction but in the opposite sense.

305. An aeroplane leaves a town of latitude 1°S, flies x km due South, then x km due East, then x km due North. He is then 3x km due East of his starting point. Find x.

306. I post a letter to a friend. There is a probability of 4/5 that a letter will reach its destination. If he received the letter he would send me a reply. What is the probability that he received the letter if I receive no reply.

307. I have 5 balls, identical in appearance, of which two are unequal in weight, one heavier and one lighter than each of the other 3. Together these 2 are equal in weight to two regular balls. Show how to distinguish the balls in three comparisons using a beam balance.

308. Seven towns $T_1, T_2, \ldots T_7$ are connected by a network of 21 one-way roads such that exactly one road runs directly between any 2 towns. (For example, the towns could be situated at the vertices of a convex heptagon, the seven sides and 14 diagonals of which form the network of roads). Given any pair of towns T_i, T_j $(1 \leqslant i < j \leqslant 7)$ there is a third town, T_k, such that T_k can be reached by a direct route from both T_i and T_j.

(i) Prove that of the 6 roads with an end at any town T_i, the number in which traffic is directed away from T_i is at least 3. Hence prove that it is exactly 3.

(ii) Let the towns which can be reached directly from T_1 be numbered T_2, T_3, T_4. Show that the roads between T_2, T_3, T_4 form a circuit.

(iii) Display on a sketch a possible orientation of traffic on the 21 roads.

Solutions to Problems 285–296 (Vol. 11 No. 3)

285. C.F. Gauss was given the problem of summing the numbers from 1 to 100 when he was a student. He did it this way:–

$$S = 1 + 2 + 3 + \ldots + 100, \text{ then}$$
$$S = 100 + 99 + 98 + \ldots + 1, \text{ so}$$
$$2S = 101 + 101 + 101 + \ldots + 101 = 100 \times 101, \text{ or } S = 5{,}050.$$

We pose the harder problem of summing all the numbers from 1 to 300 which are multiples of 3 or 5 or 7. That is, calculate

$$S = 3 + 5 + 6 + 7 + 9 + 10 + 12 + 14 + 15 + 18 + \ldots + 300.$$