306. I post a letter to a friend. There is a probability of 4/5 that a letter will reach its destination. If he received the letter he would send me a reply. What is the probability that he received the letter if I receive no reply.

307. I have 5 balls, identical in appearance, of which two are unequal in weight, one heavier and one lighter than each of the other 3. Together these 2 are equal in weight to two regular balls. Show how to distinguish the balls in three comparisons using a beam balance.

308. Seven towns $T_1, T_2, \ldots T_7$ are connected by a network of 21 one-way roads such that exactly one road runs directly between any 2 towns. (For example, the towns could be situated at the vertices of a convex heptagon, the seven sides and 14 diagonals of which form the network of roads). Given any pair of towns T_i, T_j $(1 \leqslant i < j \leqslant 7)$ there is a third town, T_k, such that T_k can be reached by a direct route from both T_i and T_j.

(i) Prove that of the 6 roads with an end at any town T_i, the number in which traffic is directed away from T_i is at least 3. Hence prove that it is exactly 3.

(ii) Let the towns which can be reached directly from T_1 be numbered T_2, T_3, T_4. Show that the roads between T_2, T_3, T_4 form a circuit.

(iii) Display on a sketch a possible orientation of traffic on the 21 roads.

Solutions to Problems 285–296 (Vol. 11 No. 3)

285. C.F. Gauss was given the problem of summing the numbers from 1 to 100 when he was a student. He did it this way:–

$$S = 1 + 2 + 3 + \ldots + 100, \text{ then}$$
$$S = 100 + 99 + 98 + \ldots + 1, \text{ so}$$
$$2S = 101 + 101 + 101 + \ldots + 101 = 100 \times 101, \text{ or } S = 5{,}050.$$

We pose the harder problem of summing all the numbers from 1 to 300 which are multiples of 3 or 5 or 7. That is, calculate

$$S = 3 + 5 + 6 + 7 + 9 + 10 + 12 + 14 + 15 + 18 + \ldots + 300.$$

Answer: The sum of all multiples of 3 up to 300 is $S_3 = 3(1+2+3+\ldots+100) = \frac{1}{2} \times 3 \times 100 \times 101$. Similarly the sum of all multiples of 5 is

$S_5 = 5(1+2+3\ldots+60) = \frac{1}{2} \times 5 \times 60 \times 61$ and all multiples of 7 is
$S_7 = 7(1+2+\ldots+42) = \frac{1}{2} \times 7 \times 42 \times 43$.

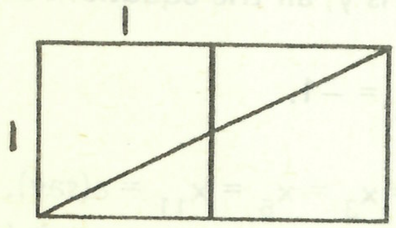
However the desired sum S is not exactly equal to $S_3 + S_5 + S_7$; for example all multiples of 15 have appeared in both S_3 and S_5, etc. We must therefore

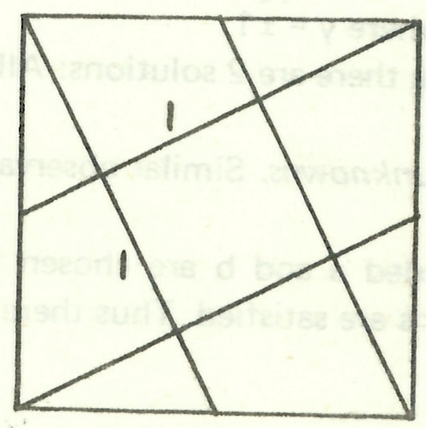
subtract $S_{15} = 15(1+2+\ldots+20) = \frac{1}{2} \times 15 \times 20 \times 21$
and similarly $S_{21} = 21(1+2+\ldots+14) = \frac{1}{2} \times 21 \times 14 \times 15$
and $S_{35} = 35(1+2+\ldots+8) = \frac{1}{2} \times 35 \times 8 \times 9$.

But $S \neq S_3 + S_5 + S_7 - S_{15} - S_{21} - S_{35}$ since all multiples of $3 \times 5 \times 7$ (viz. 105 and 210) have been added once in all of S_3, S_5 and S_7 but subtracted once in each of S_{15}, S_{21} and S_{35}. Hence

$$S = S_3 + S_5 + S_7 - S_{15} - S_{21} - S_{35} + S_{105} = 24,321.$$

286. Show how to cut up and re-assemble five squares of side length 1 into a single square.





287. Show that $\sqrt{(1976^{1977} + 1978^{1979})}$ is irrational.

Answer: Note that if m/n is a rational number in lowest terms (i.e. m and n are integers having no common factor except 1), its square is m^2/n^2 which is still in lowest terms. Hence m^2/n^2 can be an integer only if n is 1. That is, the square root of an integer, if rational, must itself be an integer; or, yet again, amongst the positive integers only the perfect squares 1, 4, 9, 16, . . . m^2, . . . have rational square roots.

Also note that if the factorisation of m into primes is $m = p_1 . p_2 . . . p^k$ then $m^2 = p_1{}^2 . p_2{}^2 . . . . p_k{}^2$ so that every prime which is a factor of m^2 divides it an even number of times. We can now see that $1976^{1977} + 1978^{1979}$ is not a perfect square since $1976^{1977} + 1978^{1979} = 2^{1979}(988^{1975} \times 499^2 + 989^{1975})$ where the number in brackets is odd; that is the prime factor 2 occurs an odd number of times.

The result is now clear.

288. Find *all* solutions of the equations with 13 unknowns:

$$x_1x_2 = x_2x_3 = x_3x_4 = \ldots = x_{12}x_{13} = x_{13}x_1 = 1.$$

Solve similar equations for 12 unknowns.

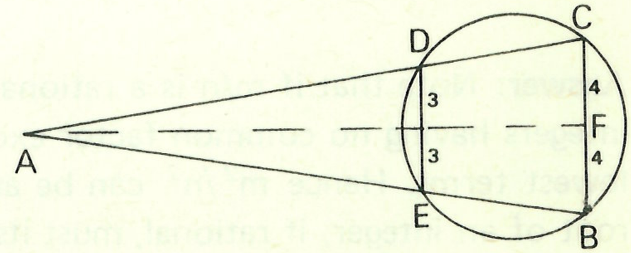
Answer: *Thirteen unknowns.* It is obvious that $x_k \neq 0$ for any k. Hence x_2 can be cancelled from $x_1x_2 = x_2x_3$ giving $x_1 = x_3$. Similarly $x_1 = x_3 = x_5 = \ldots = x_{13} = x_2 = x_4 \ldots = x_{12}$. If each of these equal numbers is y, all the equations become $y^2 = 1$ where $y = \pm 1$.

Hence there are 2 solutions: All $x_k = +1$; or all $x_k = -1$.

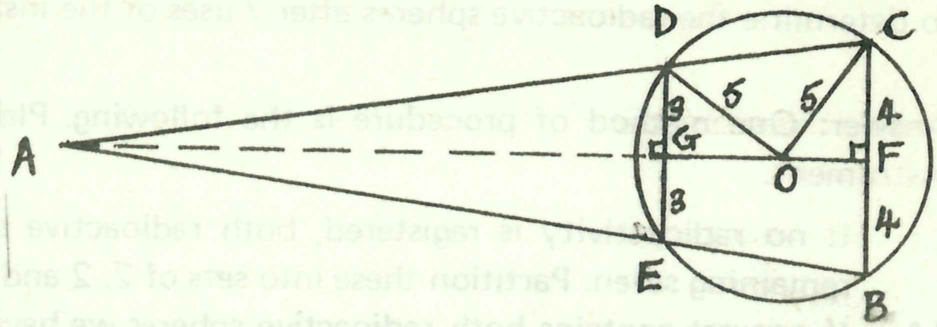
Twelve unknowns. Similar observations yield $x_1 = x_3 = x_5 = x_{11} = a$(say).
and $x_2 = x_4 = x_6 = \ldots x_{12} = b$ (say).

Provided a and b are chosen to be any two numbers whose product is 1 all equations are satisfied. Thus there are infinitely many solutions.

289. In a circle of radius 5 cm we have two parallel chords CB and ED of lengths 8 cm and 6 cm respectively. CD and EB are extended to meet at A. Calculate the altitude of the triangle ABC: that is, the length of AF.



Answer: Let O be the centre of the circle, G the mid point of DE. Then using Pythagoras' theorem in ΔOFC and ΔOGD we easily find $OF^* = 3$ cm and $OG^* = 4$ cm. From similar triangles ΔAGD and ΔAFC we have $GD^*/FC^* = AG^*/AF^* = (AF^*-FG^*)/AF^*$.



This gives $3/4 = (AF^*-7)/AF^*$, whence $AF^* = 28$ cm.

Comment: The above working assumes that O lies between F and G, which is clearly the case in the diagram accompanying the problem. You will have no difficulty in drawing a figure in which the parallel chords CB and ED are both on the same side of O. For this figure $FG^* = OG^*-OF^* = 4-3$ cm $= 1$ cm (instead of 7 cm) and the same argument with similar triangles gives $AF^* = 4$ cm.

290. If x_1, x_2, x_3, x_4, and x_5 are all positive numbers, prove that

$$(x_1 + x_2 + x_3 + x_4 + x_5)^2 \geqslant 4(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1).$$

Answer: Since both sides of the inequality are left unchanged if we move all subscripts to the right by the same amount (bringing any subscript to the beginning whenever it reaches the end) we may assume that the smallest number is x_1.

Now since we want an expression with terms x_1x_2, x_2x_3, x_3x_4 and x_4x_5, we look at the expression $(x_1-x_2+x_3-x_4+x_5)^2$, where the signs alternate. Since this is a perfect square,

$$\begin{aligned} 0 &\leqslant (x_1-x_2+x_3-x_4+x_5)^2 \\ &= (x_1+x_2+x_3+x_4+x_5)^2 - 4(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1) \\ &\quad - 4(x_1x_4+x_2x_5-x_5x_1) \end{aligned}$$

$$\begin{aligned} \text{So } (x_1+x_2+x_3+x_4+x_5)^2 - 4(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1) &\geqslant 4(x_1x_4+x_2x_5-x_5x_1) \\ &= 4x_1x_4 + 4x_5(x_2-x_1) \\ &\geqslant 0 \end{aligned}$$

since $x_2 \geqslant x_1 \geqslant 0$ and $x_4, x_5 \geqslant 0$.

291. Among 11 apparently identical metal spheres, 2 are radioactive. We have an instrument which detects the presence of radioactivity. Show that it is possible to determine the radioactive spheres after 7 uses of the instrument.

Answer: One method of procedure is the following. Place any 4 spheres in the instrument.

1. If no radioactivity is registered, both radioactive spheres are amongst the remaining seven. Partition these into sets of 2, 2 and 3 and test each set.

1A. If one set contains both radioactive spheres we have already finished unless it is the set of 3; in this case after testing at most 2 individual spheres the situation will be clear.

1B. On the other hand, if two of the sets are radioactive each contains one radioactive sphere. To take the most difficult case, assume that one of the set of 3 is radioactive. Test two individual spheres from this set. We still have the seventh use of the instrument which enables us to find the radioactive ball in the relevant set of 2. (Test either ball)

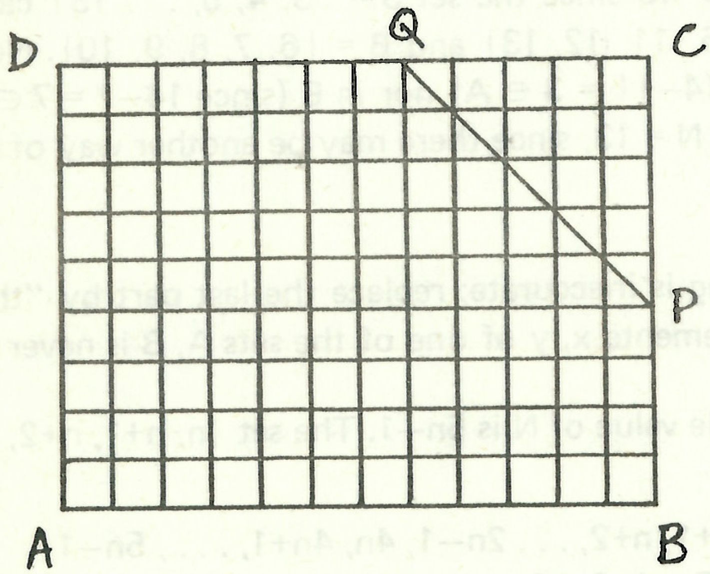
2. If the chosen set of 4 spheres show radioactivity, replace them by the remaining 7. We may assume that this second use of the instrument shows radioactivity (implying one radioactive sphere in the set of 4, and one in the set of 7), since otherwise both radioactive balls are amongst the 4, and we have plenty of uses of the instrument left to check them one at a time. Now the one radioactive ball in the set of 4 can be located in 2 measurements, and the one in the set of 7 in 3 measurements. The process is to divide the set into two halves (as nearly as possible), and decide which half contains the radioactive sphere by testing either half.

292. The plane is divided (like a chessboard) into congruent squares. A finite number of squares are coloured black, the others (infinitely many) remain white. After 1 second, the squares change their colour according to the following rule: if the upper and right neighbours of a given square, S, have the same colour then S takes this colour (irrespective whether it had this colour already or not); if they have opposite colours then the colour of S remains unchanged. This process is repeated after 2 seconds, 3 seconds, ... Describe the eventual colouring of all squares, and prove your assertion.

Answer: Since there are only a finite number of black squares at the start one can find a rectangle which contains all of them. Suppose that all squares outside ABCD are initially white. It is easy to check that every square outside ABCD remains white after 1 second, and therefore, by the same check, indefinitely. Note that the top right hand square (at corner C) in the rectangle will certainly be white after 1 sec, whatever colour it was initially.

Now suppose that all squares completely above and to the right of some line PQ become white after n secs, where $\angle PQC = 45°$ (see figure). Then after $(n + 1)$ secs they remain all white, and in addition the squares *on* the line PQ will also all be white.

It follows that white squares "invade" the rectangle ABCD at the top right hand corner and advance relentlessly at least one "diagonal" every second until, when the diagonal reaches the corner square at A, every square has become white. The maximum time taken is $(l + b-1)$ secs where l and b are the numbers of squares in the length and breadth respectively of ABCD.



293. Determine all positive integers n such that $1! + 2! + 3! + \ldots + n!$ is a perfect square.

Answer: Let $S_n = 1! + 2! + 3! + \ldots + n!$
Then S_1 (= 1) and $S_3 = 9$ are both perfect squares.

Since, if $n \geqslant 5$ $n! = 1.2.3 \ldots 5. \ldots, n!$ is divisible by 10. It follows that S_{n-1} and S_n, when written in usual decimal notation both end with the same digit for $n \geqslant 5$. Hence $S_4, S_5, S_6, \ldots \ldots$ all end with the same digit; viz 3, since $S_4 = 33$.

But all perfect squares end with one of the digits 0, 1, 4, 9, 6 or 5. Hence S_n is never a perfect square for $n \geqslant 4$.

294. Given a positive integer n, find (in terms of n) the largest integer N such that the set of integers $S = \{n, n+1, n+2, \ldots N\}$ can be split up into two subsets A and B such that $A \cup B = S$ and the difference $x-y$ between any two elements x,y of one of the sets A,B is in the other set.

E.g. if $n = 3$, $N \geqslant 13$ since the set $S = \{3, 4, 5, \ldots 13\}$ can be split into the subsets $A = \{3, 4, 5, 11, 12, 13\}$ and $B = \{6, 7, 8, 9, 10\}$. Note that 14 cannot be put in A (since $14-11 = 3 \in A$) nor in B (since $14-7 = 7 \in B$). However, this does not prove that N = 13, since there may be another way of choosing A and B.

Answer: The wording is inaccurate; replace the last part by "the difference $x-y$ between any two elements x, y of one of the sets A, B is never in that same set."

The largest possible value of N is $5n-1$. The set $\{n, n+1, n+2, \ldots, 5n-1\}$ can be partitioned into

$$A \equiv \{n, n+1, n+2, \ldots 2n-1, 4n, 4n+1, \ldots, 5n-1\}$$

and

$$B \equiv \{2n, 2n+1, 2n+2, \ldots, 4n-1\}$$

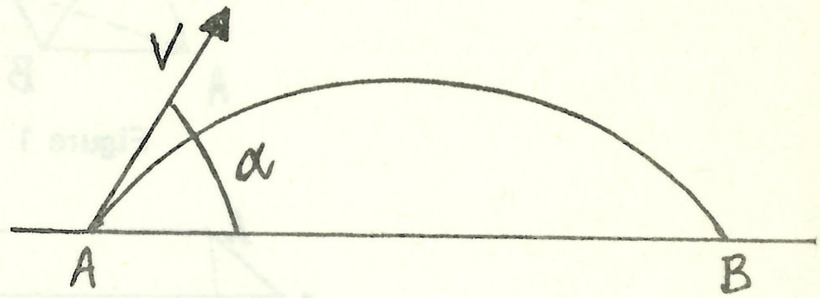
subject to the required condition. Hence $N \geqslant 5n-1$. However it is impossible to appropriately partition the set $n, n+1, n+2, \ldots, 5n$ since the five numbers n, 2n, 3n, 4n and 5n cannot be apportioned to sets A and B. Thus if $n \in A$ we must have $2n \in B$, and then $4n \in A$, and then $3n \in B$. But now it is impossible to place 5n in either set.

295. *(Submitted by Mr J. Scott of Barker College)* A man is able to throw a cricket ball 30 metres vertically upwards. What is the furthest distance he can throw it horizontally? (Ignore any air resistance).

Answer: Let V metres/sec. be the velocity with which he can throw the ball. Then equating its kinetic energy $\frac{1}{2}mV^2$ with its gravitational potential energy mg.30 at a height of 30 metres yields.

$V^2 = 60$ g. (g metres/sec^2 is the acceleration due to gravity).

Now suppose he throws the ball with velocity V at an elevation of α to the horizontal. We are interested in the horizontal range, AB*.



The vertical component of velocity is V sin α upwards at A, and $-$V sin α at B. The time elapsing is therefore (2v sinα)g secs. (since the vertical acceleration is equal to g throughout). Hence horizontal distance travelled

= horizontal component of velocity x time of flight.
= V cos α x (2V sin α)/g = (V^2 sin 2α)/g = 60 sin 2α metres.

Hence the greatest horizontal range achievable is 60 metres by throwing the ball at an angle of 45° to the horizontal.

NOTE: The answer to this question does not depend either on the mass of the ball or the value of g.

296. A parallelepiped is a solid figure with six faces each of which is a parallelogram. You are given four points, A, B, C, D in space not all lying in the same plane. How many parallelepipeds exist with A, B, C, D included amongst the eight vertices?

Answer: The four points A, B, C, D may be thought of as the vertices of a tetrahedron (Figure 1) and so, for the arbitrary parallelepiped in Figure 2, we consider the possible tetrahedra obtained by choosing 4 of its vertices not all in the same plane. These tetrahedra can be put into 4 classes as follows.

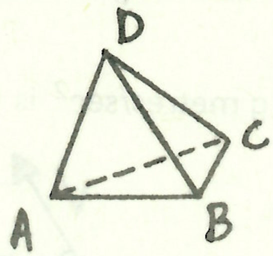


Figure 1

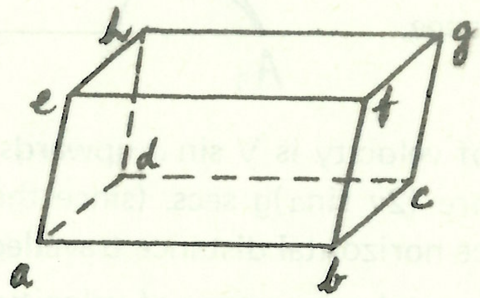


Figure 2

Type 1: Example a b d e. Three edges of the tetrahedron are three concurrent sides of the parallelepiped. There are 4 ways in which the given tetrahedron may be so viewed – the three edges meeting at any one of its 4 vertices being taken as edges of the parallelepiped.

Type 2: Example a b c g. Three edges of the tetrahedron are edges of the parallelepiped, but two of them (ab and cg) do not intersect.

There are 12 ways of so viewing the given tetrahedron. There are 3 pairs of opposite edges in the tetrahedron. Once one of them (e.g. AD and BC) has been selected there are 4 other edges from which to select another side of the parallelepiped.

Type 3: Example a b e g. Two edges of the tetrahedron are also edges of the parallelepiped. The edges of one triangular face eb, bg and ge, are face diagonals of the parallelepiped, the sixth edge, ag, joining opposite vertices of the parallelepiped.

There are 12 ways of so assigning the sides of the given tetrahedron: first there are 4 ways of choosing a face whose edges are to be face diagonals, and once one (e.g. ABC) has been chosen there are 3 ways of choosing the diagonal through the parallelepiped (DA, DB, or DC).

Type 4: Example a c f h. All 6 sides of this tetrahedron are face diagonals of the parallelepiped.

Since no choice is involved for the sides, there is only one way of viewing the given tetrahedron as a type 4 tetrahedron.

It is not difficult to check that *every* tetrahedron inscribed in the parallelepiped is of one of the four types. It is clear also that once the given tetrahedron has been viewed in one of the 4 types of ways, a unique parallelepiped is determined. (This is almost immediately evident for Types 1, 2 and 3. For Type 4 observe that the joins of mid-points of opposite sides of the tetrahedron are line segments equal in length and parallel to the sides of the parallelepiped.)

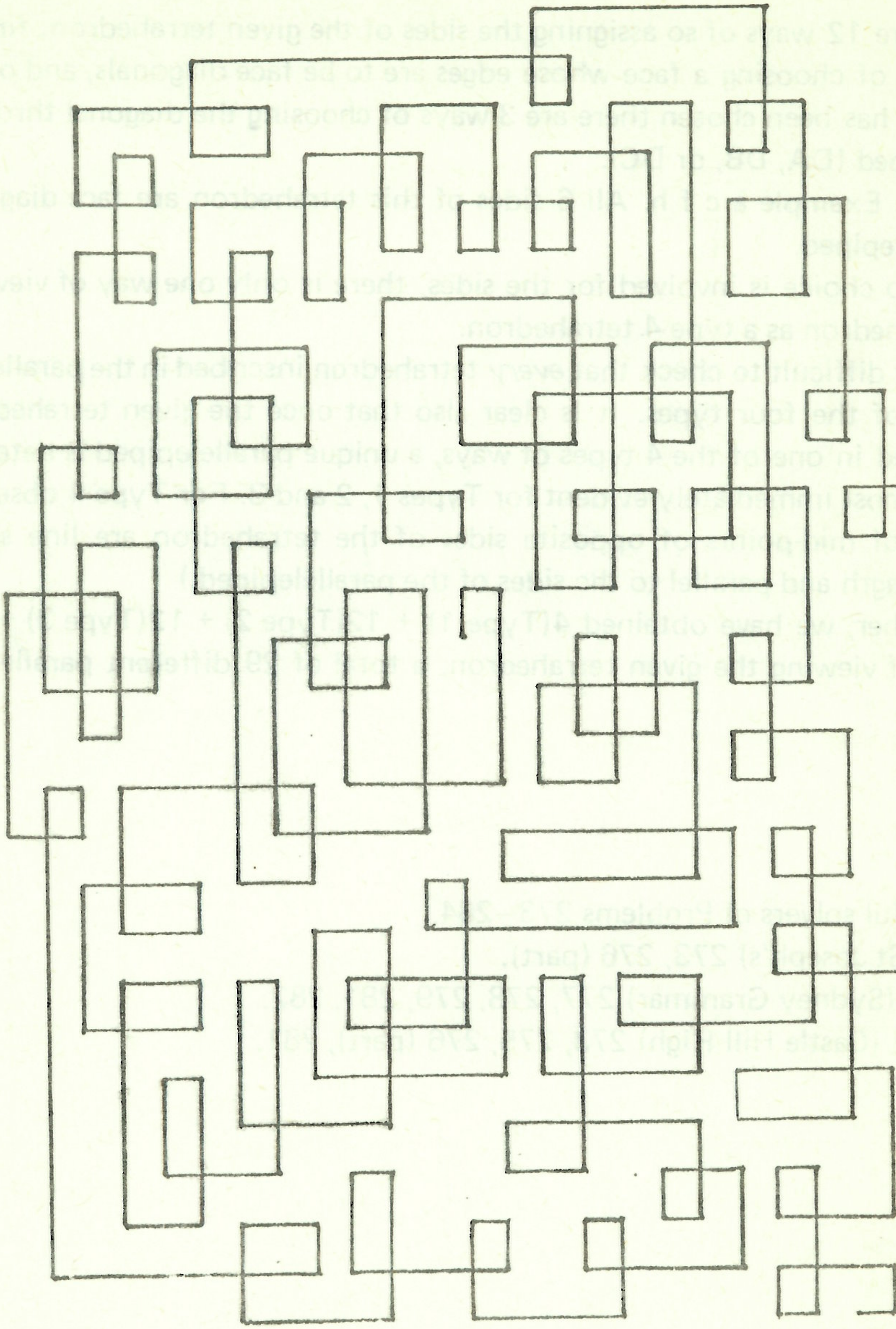
Altogether, we have obtained 4(Type 1) + 12(Type 2) + 12(Type 3) + 1(Type 4) ways of viewing the given tetrahedron, a total of 29 different parallelepipeds resulting.

Successful solvers of Problems 273–284.

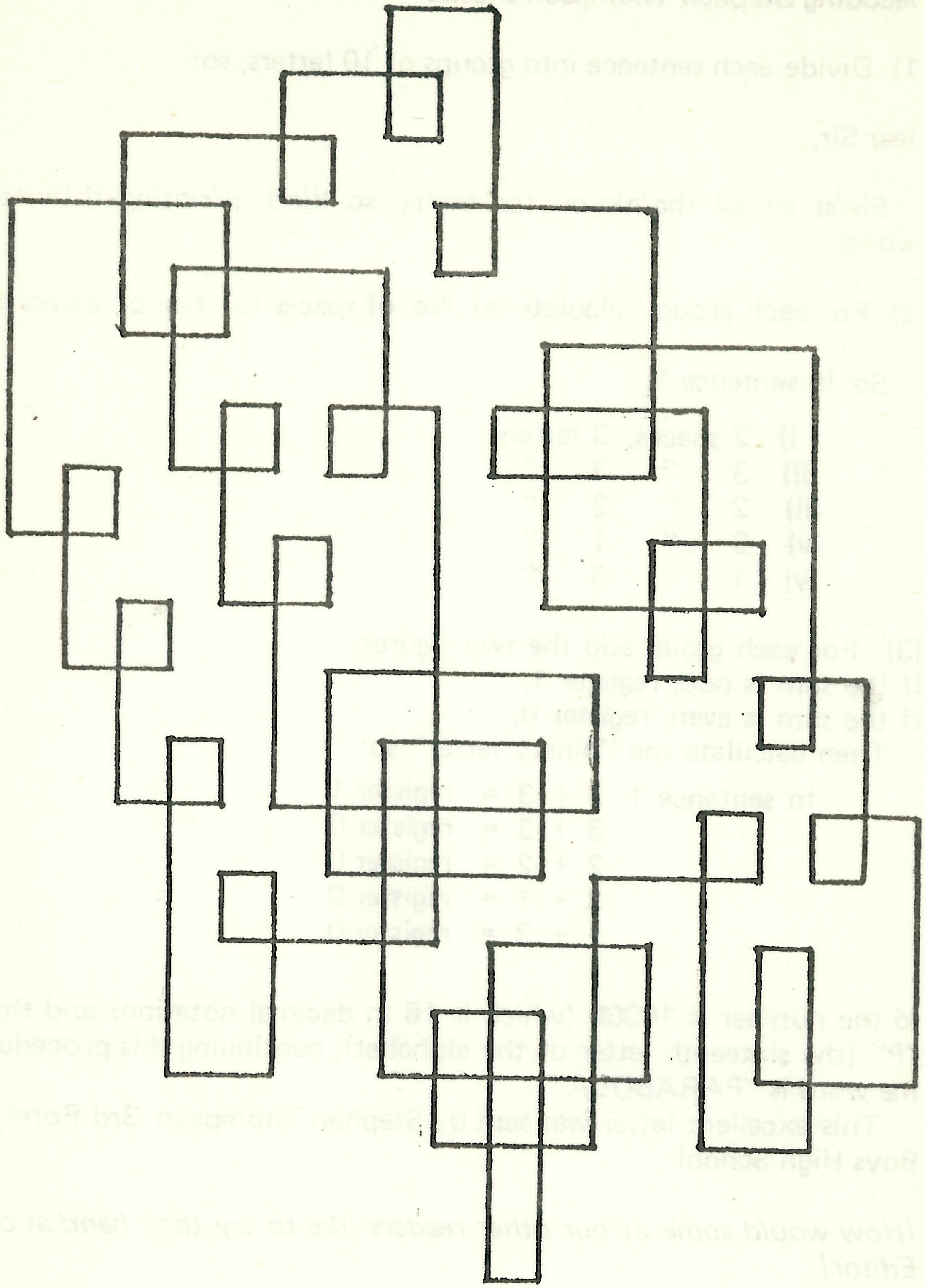
G. Clark (St Joseph's) 273, 276 (part).

A. Fekete (Sydney Grammar) 277, 278, 279, 281, 282.

M. Hartley (Castle Hill High) 273, 275, 276 (part), 281.



See page 15



See page 15

)ecoding Stephen Thompson's letter

1) Divide each sentence into groups of 10 letters, so:

)ear Sir,

Fir/st of all tha/nkyou for be/ing so kind, p/rinting thi/~~s~~ ~~letter~~ (disregard ·xtras).

2) For each group, calculate (a) No. of spaces (b) No. of letters after last space.

So, in sentence 1:

i)	2 spaces,	3 letters
ii)	3 "	3 "
iii)	2 "	2 "
iv)	3 "	1 "
v)	1 "	3 "

(3) For each group add the two figures.
If the sum is odd, register 1.
If the sum is even, register 0.
Then calculate the "binary letter" so:

In sentence 1:
$2 + 3 =$ register 1
$3 + 3 =$ register 0
$2 + 2 =$ register 0
$3 + 1 =$ register 0
$1 + 3 =$ register 0

so the number is 10000 (which is 16 in decimal notation) and thus, the letter is "P" (the sixteenth letter of the alphabet); continuing this procedure we find that the word is "PARABOLA".

This excellent letter was sent by Stephen Thompson, 3rd Form, North Sydney Boys High School.

[How would some of our other readers like to try their hand at coded letters? – Editor]