MEASURE FOR MEASURE

Problem 297 in the last issue of Parabola is an example of a very popular type of mathematical puzzle in which you are asked to measure a certain quantity of liquid using only a set of jugs. The liquid must be transferred from jug to jug either filling or emptying a jug at each stage. Is there a more mathematical way of solving these puzzles than just trial and error? By looking at some examples we will try to find some general methods.

Problem 1. You have a 6 litre jug and an 8 litre jug and you wish to get 1 litre of water from a river (from which you can take as much water as you like, or into which you can empty as much water as you like).
Answer. Since the jugs hold 6 and 8 litres, we can only transfer an even number of litres at a time. Since we started with no water, we must end up with an even number of litres and so the problem cannot be solved. Similarly, we cannot measure out any odd number of litres with these two jugs.

Problem 2. Measure out 1 litre of water from the above river with a 5 litre jug and a 7 litre jug.
Answer. Note that $1 = 3 \times 5 - 2 \times 7$, and so we need to fill the 5 litre jug three times and empty the 7 litre jug twice. To illustrate this, we will use an ordered pair (x,y) of two integers x and y where the smaller jug has x litres and the larger jug has y litres. Starting with both jugs empty, we use the following steps:

$$(0,0) \rightarrow (5,0) \rightarrow (0,5) \rightarrow (5,5) \rightarrow (3,7) \rightarrow (3,0) \rightarrow (0,3) \rightarrow (5,3) \rightarrow (1,7) \rightarrow (1,0)$$

where the first, third and seventh arrows mean "fill the 5 litre jug", the fifth and last arrows mean "empty the 7 litre jug" and the other arrows mean "transfer as much water as possible from the 5 litre jug to the 7 litre jug".

Problem 3. Using the equipment in problem 2, measure out 2 litres, 3 litres, . . . 12 litres.
Answer. Doubling the equation in problem 2, we get $2 = 6 \times 5 - 4 \times 7$. Subtracting $0 = 7 \times 5 - 5 \times 7$, we get $1 = 1 \times 7 - 1 \times 5$ and so a solution is $(0,0) \rightarrow (0,7) \rightarrow (5,2) \rightarrow (0,2)$. Similarly, we can see that $3 = 2 \times 5 - 1 \times 7$ and so a solution for 3 litres is $(0,0) \rightarrow (5,0) \rightarrow (0,5) \rightarrow (5,5) \rightarrow (3,7) \rightarrow (3,0)$. The rest are up to you.

In the same way, if d is the largest positive integer which divides both the positive integers a and b, then we can find positive integers x and y such that d = ax–by (ask your teacher how to do it!). Using this equation, or any multiple of it, we can find a way to measure any multiple of d litres from the river using jugs which hold a litres and b litres. For example, we can get 4 litres in problem 1 by using 4 = 2 x 6–1 x 8.

Problem 4. Solve problem 297 from Vol. 12 No. 1.
Answer. Instead of solving this problem, we will first work out how to measure 4 litres of water using a 3 litre jug and a 5 litre jug from a river. From the equation 4 = 3 x 3–1 x 5, we get the solution

(0,0) → (3,0) → (0,3) → (3,3) → (1,5) → (1,0) → (0,1) → (3,1) → (0,4).

Now we notice that at no time do we take more than 8 litres from the river nor put more than 8 litres into the river, and so we may pretend that the river is an 8 litre jug. Thus, a solution to problem 297 may be found by using triples (x,y,z) where x and y are the numbers above and z is the amount remaining in the initially full 8 litre jug called "river":

(0,0,8) → (3,0,5) → (0,3,5) → (3,3,2) → (1,5,2) → (1,0,7) → (0,1,7) → (3,1,4) → (0,4,4).

Problem 5. Given a full 8 litre jug of water, an empty 3 litre jug and an empty 7 litre jug, measure 2 litres into the smallest jug.
Answer. In order to have 2 litres in the smallest jug, the other 6 litres must be in either of the other 2 jugs, i.e. (2,6,0) or (2,0,6). This time the largest jug will only hold 8 litres which is less than the total of the other two jugs. As our "river solution" might create difficulties (e.g. if it requires us to have 2 litres in the smallest jug and the 7 litre jug full – a total of 9 litres) we cannot use this method.

A graphical method of doing this was described in 1939 by Mr M. Tweedie. Draw an equilateral triangle with each side 8 cm long (representing the total of 8 litres of water), and divide each side up into 1 cm intervals with dots. By joining these dots with lines parallel to the sides of the triangle we may now divide the large triangle into 28 smaller triangles with 1 cm sides (see Fig. 1). Each dot at the vertex of a small triangle represents one of the ordered triples (x,y,z) where x is the number of litres in the smallest jug, y is the number of litres in the medium jug and z is the number of litres in the largest jug (see problem 4). The dots along the base of the large triangle represent the ordered triples (x,y,0) which say that the largest jug is empty, the dots on the next line up (parallel to the base) represent the triples (x,y,1) in which the largest jug has 1 litre and so on. For example, the apex of the large triangle will be labelled (0,0,8) and so represents the fact that the largest jug is full and the other two jugs are empty. Similarly the left hand side represents the fact that the smallest jug is empty and the lines parallel to it represent the fact that the smallest jug has 0,1,2, . . . , 8 litres; also the lines parallel to the right hand side represent the fact that the medium jug has 0,1,2, . . . , 8 litres. Some examples are shown in Fig. 1, and you should try to work the rest out for yourself.

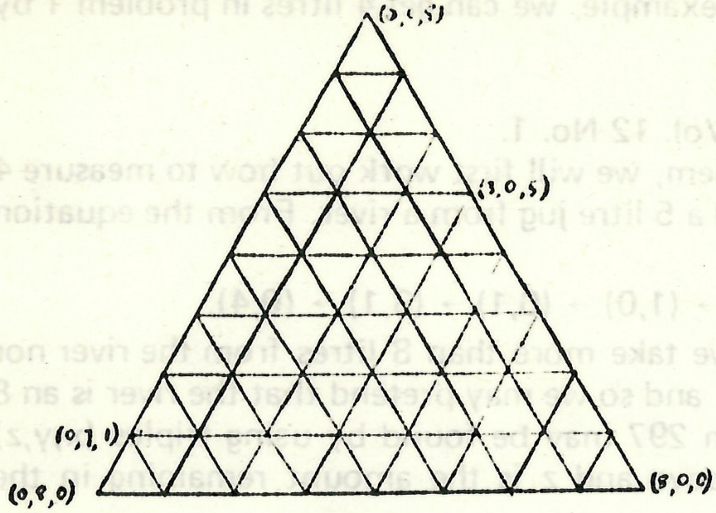

Figure 1

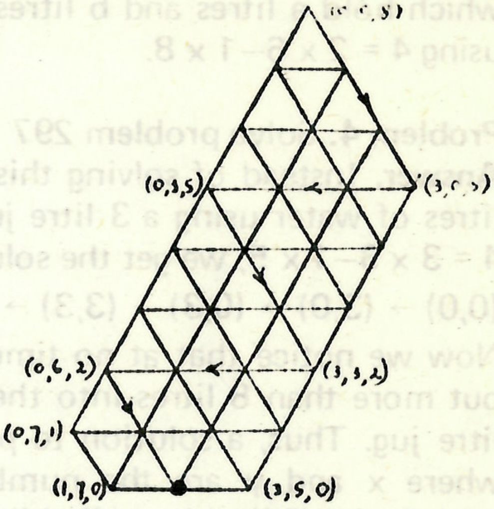

Figure 2

However, when we look at some of the dots, we see that they represent impossible situations. For example, (8,0,0) says that the smallest jug has 8 litres which is impossible. In fact, this jug will only hold 3 litres and so we should erase all the dots to the right of the line representing the fact that it has 3 litres. Similarly, the dot (0,8,0) should be erased, leaving Figure 2.

At last, we are in a position to answer problem 5. It is not hard to see that filling or emptying a jug is represented by one of the lines of Figure 2 which joins a dot on an outside edge to a dot on a different outside edge of the figure, such as (0,0,8) → (3,0,5) or (3,0,5) → (0,3,5). Note that we may suppose (0,0,8) is on the edge representing the smallest jug being empty and so (3,0,5) is on a *different* edge. By working backwards from the dot (2,6,0) to the dot (0,0,8) we can find the following path which solves the problem:

(0,0,8) → (3,0,5) → (0,3,5) → (3,3,2) → (0,6,2) → (2,6,0).

Problem 6. Find a solution to problem 297 which uses the fewest number of steps possible.

Answer. We will use the graphical method again. As we did in the previous problem, we can represent the contents of the three jugs by Figure 3. To find a quickest solution, we label the starting point with 0. Any points which we can reach by going along a line to a different edge will be labelled 1, as in Fig. 4 (remember that (0,0,8) may be thought of as being on the left hand edge of the original large triangle and so (3,0,5) is on a different edge; similarly (0,0,8) may be thought of as being on the edge with 0 as the middle of the ordered triple and so (0,5,3) is on a different edge this time). We now label with 2 any points we can reach from a point labelled 1 by this process unless it has already been labelled (thus (0,0,8) will not be labelled 2 as it is already labelled 0). We continue using this process of labelling as shown in Fig. 4 until we reach the point (0,4,4), which is the point we are looking for. To find a shortest solution, all we have to do is find a path from (0,0,8) to (0,4,4) which does not pass through two points with the same label. One such path is:

(0,0,8) → (0,5,3) → (3,2,3) → (0,2,6) → (2,0,6) → (2,5,1) → (3,4,1) → (0,4,4).

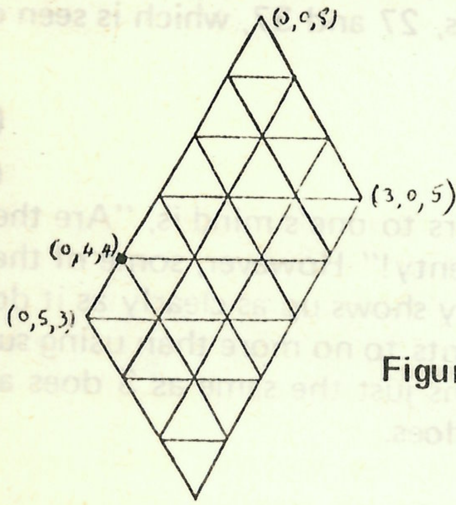

Figure 3

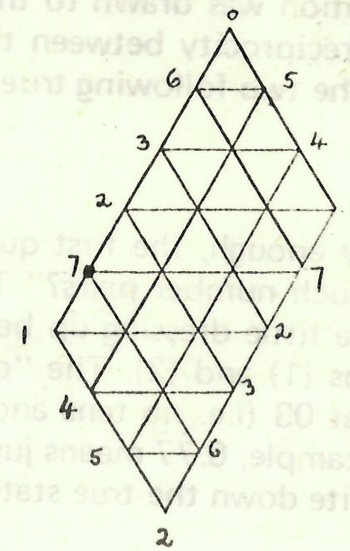

Figure 4

Some problems for you to try

(7) Devise a method for obtaining an infinite number of solutions to problem 2 (Hint: 7 x 5–5 x 7 = 0 and so can be added to the equation 3 x 5–2 x 7 = 1). Use this method to find solutions to problem 3 with the least number of steps possible.

(8) You are given a 24 litre jug full of water and three empty jugs which will hold exactly 5 litres, 11 litres and 13 litres. Using no other means of measurement, divide the water into three equal amounts of 8 litres in the least number of steps possible (Hint: Use a tetrahedron with some of the vertices cut off as in problems 5 and 6).

(9) You are given two empty jugs which hold exactly 5 litres and 7 litres, and a barrel containing some water (which cannot be refilled from the jugs). You may only fill the jugs from the barrel or from each other, but you may empty the jugs onto the ground at any time. What is the least amount of water you will need in order to measure 1 litre into each jug? (Hint: The last operation must be pouring the 1 remaining litre from the barrel into one of the jugs.)

Some books to read

Scientific American, September 1963.

"Puzzles and Paradoxes" by T. O'Beirne (Oxford University Press, 1965).