Parabola Volume 12, Issue 2 (1976)

YOUR LETTERS

Dear Sir,

Other readers of Parabola might be interested in the following mathematical trick:

Ask a friend to think of a polynomial $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ where the co-efficients $a_0, a_1, \ldots, a_n$ are all positive integers. Now ask him to choose an integer x_0 greater than all the co-efficients and work out the value of $f(x_0)$. If your friend gives you the values of x_0 and $f(x_0)$, you can now work out the polynomial that he picked.

You would probably say that it is impossible, but it is quite easy.

Graeme Elsworthy,
Birrong Boys' High.

[Graeme's explanation is given on page 18 – Editor]

Dear Sir,

I think I have discovered a formula for prime numbers, the problem is I can't prove or disprove it. So, upon reading your magazine, I decided to write to you to see if you could throw some light on the matter.

The formula is $p! + 2^p - 1$ where p is a prime number.

Substituting for p,

$p = 2$ yields $2! + 2^2 - 1 = 5$ is prime

$p = 3$ yields $3! + 2^3 - 1 = 13$ is prime

$p = 5$ yields $5! + 2^5 - 1 = 151$ is prime

$p = 7$ yields $7! + 2^7 - 1 = 5167$ is prime, etc.

I would be very grateful to anybody who could either prove or disprove my formula or help in any other way.

Danny Zulaikha,
Moriah College.

Editor's reply: In deciding whether $p! + 2^p - 1$ is a prime number, we have to look for factors of this number, and clearly we only need to find a prime q which divides $p! + 2^p - 1$ (but is not equal to it). To do this, I have made use of two theorems about the prime q:

$$(q-1)! \equiv -1 \pmod q$$

and

$$2^{q-1} \equiv 1 \pmod q$$

where "$a \equiv b \pmod q$" means that q is a factor of $a - b$.

Let us look for two primes p,q where $q = p + 4$ and $p! + 2^p \equiv 1 \pmod q$ — i.e. q is a factor of $p! + 2^p - 1$.

Since $p = q - 4$, the first theorem above tells us that

$$(q-1)(q-2)(q-3)p! = (q-1)! \equiv -1 \pmod q$$

and so

$$8(q-1)(q-2)(q-3)p! \equiv -8 \pmod q.$$

The second theorem tells us that $8 \times 2^p = 8 \times 2^{q-4} = 2^{q-1} \equiv 1 \pmod q$ and so

$$8(q-1)(q-2)(q-3)2^p \equiv q^3 - 6q^2 + 11q - 6 \pmod q \equiv -6 \pmod q.$$

Adding these two equations, we have

$$8(q-1)(q-2)(q-3)(p! + 2^p) \equiv -14 \pmod q.$$

Since $8(q-1)(q-2)(q-3) \equiv -48 \pmod q$, the only way in which we can have $p! + 2^p \equiv 1 \pmod q$ is if $14 \equiv 48 \pmod q$. That is, q divides 34, and so q = 17. Thus 17 divides $13! + 2^{13} - 1$ and this is the only case where a *prime* number p+4 divides $p! + 2^p - 1$.

One of our readers might now like to try the case where p,q are primes and q = p + 6.

Dear Sir,

Reading the Time/Life book on Mathematics I've read that with an ordinary pack of 52 cards the total of sequences possible is a figure 68 figures long. I'd very much appreciate this solution explained?

Similar to this I've been trying the problem in the "Sun" newspaper each night where there are 8 alphabet letters, where one special letter is to be used in each word and each word is to be at least 4 letters long and an eight lettered word can be formed using each letter. How could this be done mathematically and how many arrangements are possible?

Jimmy Pike

Editor's Reply: Your first question is easy to answer. If we wish to form a sequence of 52 cards, we may choose any one of them first, giving 52 ways of choosing the first card of a sequence. Having done this there are 51 cards left to choose from and we may choose any of these second in our sequence. Continuing this argument, we can see that there are $52! = 52 \times 51 \times \ldots \times 3 \times 2 \times 1$ ways of choosing a complete sequence. To see how large this number is, add the logarithms of them: $\log 2 + \log 3 + \ldots + \log 51 + \log 52 = 67.9$, showing that there are 68 digits in the answer.

The answer to your second question is a little harder. I looked at two of the examples from the "Sun" and noticed that one of the letters (not the special letter) occurred twice, and so I have considered the case where there are 7 letters one of which occurs twice (you may like to try the case when no given letter is repeated or when one letter occurs more than twice). The first thing to notice is that it is impossible to find mathematically all the English words with (say) 4 letters as there is no way mathematically of deciding whether an arbitrary arrangement of 4 letters is an English word (you will need a dictionary for that). What follows is the *total* number of "mathematical words" (i.e. arrangements of letters whether or not they form an English word).

If we wish to form a 5-lettered "word", we consider two possibilities:

(a) If the repeated letter does not occur twice in our "word", then we must choose the special letter and 4 letters from the remaining 6 letters: the number of ways of doing this is

$$^6C_4 = 6!/(4! \times 2!) = 15.$$

The number of ways of arranging the chosen 5 letters is 5!, and so the number of "words" is $5! \times 15$.

(b) If the repeated letter occurs twice in our "word", we have already chosen 3 letters (the special letter and the repeated letter twice) and (as before) the number of ways of choosing the other 2 letters from the remaining 5 letters is $^5C_2 = 10$. The number of ways of arranging 5 letters, one of which occurs twice, is $\frac{1}{2} \times 5!$ and so the number of "words" is $5 \times 5!$.

Adding (a) and (b) together we see that the total number of possible 5 lettered "words" is $20 \times 5! = 2400$. Similarly, we get:

no. of letters =	4	5	6	7	8
no. of "words" =	540	2400	7920	17640	20160

Of course, most of these "words" are not English words – and this is where you need your dictionary!

Dear Sir,

Recently, I noticed something interesting concerning analytical geometry and complex numbers. Consider two lines,

$$y = ix + c_1 \qquad (1)$$

$$y = ix + c_2 \qquad (2)$$

These lines are parallel, because $i = i$, but because $i^2 = -1$ and the product of their gradients is thus -1, they are also perpendicular.

As they are perpendicular, we can find a solution, and it turns out that $c_1 = c_2$. So we have a single line, naturally parallel to itself, but also perpendicular to itself at every point.

Furthermore, if we find the perpendicular distance of a point (x,y) from this line $ix - y + c_1 = 0$, using the formula $d = |ax + by + c|/\sqrt{(a^2 + b^2)}$ we see the distance is undefined as $a^2 + b^2 = 0$.

Sandy Anderson,
Form 5, Canberra Grammar School.

Editor's reply: Sandy's difficulties concerning "complex geometry" have been faced by mathematicians, who have overcome them by considering complex conjugates: if x,y are real numbers, then the conjugate of the complex number $z = x + iy$ is $\bar{z} = x - iy$. Using this, two lines with gradients m_1 and m_2 are perpendicular if $m_1 m_2 = -1$ and the square of the distance between the points (x_1, y_1) and (x_2, y_2) is $(x_1 - \overline{x_2})(x_1 - x_2) + (y_1 - y_2)\overline{(y_1 - y_2)}$.

If you have any questions similar to the ones above, send them to the Editor, who will answer them in a future issue of Parabola.