SCHOOL MATHEMATICS COMPETITION 1976

Junior Division

Question 1: Alan owes Bruce 45c, and Charles owes Alan 25c. Alan has only a $1 note, Bruce has only a 5c coin, and Charles has only a 50c coin and four 10c coins.

(i) Can the debts be settled?

(ii) If Alan finds a 50c coin, can they be settled?

Answer: (i) No. Alan must end up with 80c, Bruce with 50c, Charles with 65c, all of which amounts are less than $1, so no-one could end up holding the $1 note.
(ii): Yes. Alan gives his 50c coin to Bruce, Bruce gives his 5c coin to Charles, and Charles gives Alan three 10c coins.

Question 2: Two church bells begin ringing at the same time. The strokes of one follow regularly at intervals of 1 ⅓ seconds, while the interval between the strokes of the other is 1¾ seconds. How many strokes are heard during 10 minutes if 2 strokes following each other in an interval of ½ second or less are indistinguishable?

Answer: The first bell rings every 16 twelfths of a second, the second every 21 twelfths of a second. So in every interval of 16 x 21 = 336 twelfths of a second, the first bell rings 21 times, the second bell 16 times.

Bell 1: 0, 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176,
Bell 2: 0, 21, 42, 63, 84, 105, 126, 147, 168,

Bell 1: 192, 208, 224, 240, 256, 272, 288, 304, 320, 336
Bell 2: 189, 210, 231, 252, 273, 294, 315, 336

Of these 37 separate strokes, only 24 can be distinguished, since strokes separated by 6 twelfths of a second or less are indistinguishable.
So, in 10 minutes = 600 seconds = 7200 twelfths of a second, there will be 21 cycles each of 336 twelfths of a second, followed by an interval of 144 twelfths of a second. During this period $21 \times 24 + 11 = 515$ strokes will be heard (if one includes the stroke of bell number 1 right at the end of the 10-minute period).

Question 3: Suppose a and b are positive integers. Show that $\sqrt{2}$ lies between a/b and (a + 2b)/(a + b).

Can you find a similar result involving $\sqrt{n}$?

Answer: Suppose $a/b < \sqrt{2}$.
Then, adding 1, $(a + b)/b < \sqrt{2} + 1$.
Therefore $b/(a + b) > 1/(\sqrt{2} + 1) = \sqrt{2} - 1$.
Adding 1, $(a + 2b)/(a + b) > \sqrt{2}$.
Similarly, if $a/b > \sqrt{2}$, then $(a + 2b)/(a + b) < \sqrt{2}$.
If a and b are positive integers, then $\sqrt{n}$ lies between a/b and (a + nb)/(a + b).

Question 4: At a banquet for the 50 Knights of the Round Table, all the Knights are served red or white wine. Before tucking in, all the men who were served red wine pass their goblet one place to the left, while all those who were served white wine pass theirs two places to the right. Prove that if some started with red and some started with white, then at least one of the Knights ends up with no wine.

Would this still have been true if King Arthur had also been at the table?

Answer (i): Suppose, on the contrary, that every Knight ends up with some wine. Then every third Knight must have originally had the same colour. For, suppose not. Then there were two Knights 3 places apart with different colours, thus:

R Y X W or W Y X R

In the first case, the Knight X ends up with no wine; in the second case, the Knight X ends up with two goblets of wine, so someone else goes without.

So far we know that every third Knight had the same colour. Pick any Knight. Suppose he had red wine. Then the Knights 3, 6, 9, 12 and so on places to his right also had red wine. Since there were 50 Knights, and 50 is not divisible by 3, we see that, working around the table 3 times, every Knight must have had red wine. But this contradicts the fact that at least one Knight had white wine. The conclusion is that some Knight ends up with no wine.

(ii): If King Arthur had been at the banquet, there would have been 51 men present, and 51 is divisible by 3. If the wine had been served RRWRRW . . . RRW or WWRWWR . . . WWR around the table, then everyone would have ended up with some wine.

Question 5: A square of side 8 metres (including its boundary) is to be painted using 3 colours. Prove that there must be two points of the same colour at least $\sqrt{65}$ metres apart. Show how the square may be coloured so that no 2 points of the same colour are more than $\sqrt{65}$ metres apart.

Answer: Suppose it is possible to colour the square with 3 colours (x,y,z) so that no 2 points of the same colour are as much as $\sqrt{65}$ metres apart. Then it is easy to check that the corners must be as in Fig. (1).

Now consider the points P,Q,R in Fig. (2). Since P is $\sqrt{65}$ metres away from a point coloured z, and is $\sqrt{114}$ metres away from a point coloured y, P is coloured x. Similarly Q is coloured y. Since R is $\sqrt{80}$ metres away from a point coloured z, it is coloured x or y. But it is $\sqrt{65}$ metres away from P which is coloured x, and $\sqrt{65}$ metres away from Q which is coloured y, so R is coloured neither x nor y, and we have a contradiction.

If the square is coloured as in Fig. (3), no two points of the same colour are more than $\sqrt{65}$ metres apart:

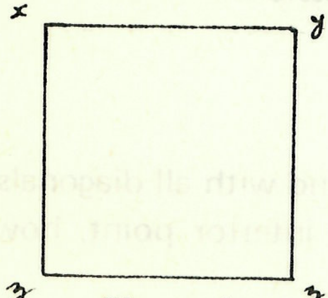

Figure 1

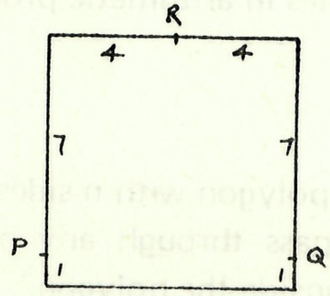

Figure 2

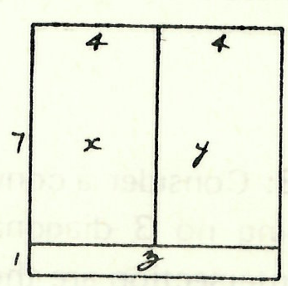

Figure 3

Senior Division

Question 1: The sum of 3 numbers in geometric progression is 21, and the sum of their squares is 189.

Find the numbers.

Answer: Let the numbers be a/r, a, ar.

Then $a/r + a + ar = 21$, $a^2/r^2 + a^2 + a^2r^2 = 189$.

Squaring the first, $a^2/r^2 + a^2 + a^2r^2 + 2(a^2/r + a^2 + a^2r) = 441$.

By subtraction, $2(a^2/r + a^2 + a^2r) = 441 - 189 = 252$,

or $a^2/r + a^2 + a^2r = 126$.

By division, $a = 126/21 = 6$,

so $6/r + 6 + 6r = 21$,

$1/r + r = 5/2$,

$r = 1/2$ or 2,

and the three numbers are 3, 6 and 12. ☆

Question 2: 5, 11, 17, 23 and 29 are five prime numbers in arithmetic progression.

Find *six* prime numbers in arithmetic progression.

Answer: If $p_1, \ldots, p_n$ are n primes in arithmetic progression, $p_1 = a$, $p_2 = a + d$, $\ldots, p_n = a + (n-1)d$, and if $p_1, \ldots, p_n$ are all greater than or equal to n, then d is divisible by every prime less than n. For if p is prime, $p < n$, $p \nmid d$, then for some k with $1 \leqslant k \leqslant n$, $a + (k-1)d$ is divisible by p. But p_k is prime, so $p_k = p$. But $p < n$, $p_k \geqslant n$, a contradiction.

So, if $p_1, \ldots p_6$ are primes in arithmetic progression, $p_1 = a, \ldots, p_6 = a + 5d$, and $p_1 \ldots p_6$ are all greater than or equal to 6, d must be divisible by 2, 3 and 5, so by 30. So the smallest set of numbers we might consider is 7, 37, 67, 97, 127, 157, and these are indeed 6 primes in arithmetic progression. ☆

Question 3: Consider a convex polygon with n sides, and with all diagonals drawn in. Assuming no 3 diagonals pass through any one interior point, how many points of intersection are there inside the polygon?

Answer: Every set of 4 vertices determines exactly one point of intersection, so the number of points of intersection is

$$^nC_4 = \binom{n}{4} = n(n-1)(n-2)(n-3)/24.$$

Question 4: Show that if $x = a_n a_{n-1} \ldots a_1 a_0$ is any natural number in decimal notation (that is, $x = a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^n a_n$), and if $S(x) = a_0 + a_1 + a_2 + \ldots + a_n$, then x and $S(x)$ leave the same remainder when divided by 9.

Find $S(S(S(1976^{1976})))$.

Answer: $x - S(x) = (a_0 + 10a_1 + \ldots + 10^n a_n) - (a_0 + a_1 + \ldots + a_n)$
$= 9a_1 + 99a_2 + \ldots + (10^n - 1)a_n$
is obviously divisible by 9, so x and $S(x)$ leave the same remainder when divided by 9. We write this as $S(x) \equiv x \pmod{9}$.

So $S(S(S(1976^{1976}))) \equiv S(S(1976^{1976})) \pmod{9}$
$\equiv S(1976^{1976}) \pmod{9}$
$\equiv 1976^{1976} \pmod{9}$

Now, $1976 \equiv 1 + 9 + 7 + 6 = 23 \equiv 2 + 3 = 5 \pmod{9}$,
$1976^2 \equiv 5^2 = 25 \equiv 7 \pmod{9}$,
$1976^3 \equiv 5 \times 7 = 35 \equiv 8 \pmod{9}$,
$1976^6 \equiv 64 \equiv 1 \pmod{9}$,
$1976^{1974} = 1976^{6 \times 329} \equiv 1^{329} \equiv 1 \pmod{9}$ and so $1976^{1976} \equiv 7 \pmod{9}$.

So $S(S(S(1976^{1976}))) \equiv 7 \pmod{9}$.

Also $1976 < 10000 = 10^4$,
so $1976^{1976} < 10^{4 \times 2000} = 10^{8000}$,
so $S(1976^{1976}) \leqslant 8000 \times 9 = 72000$,
so $S(S(1976^{1976})) \leqslant 6 + 4 \times 9 = 42$,
so $S(S(S(1976^{1976}))) \leqslant 3 + 9 = 12$.

We have $S(S(S(1976^{1976}))) \leqslant 12$ and $S(S(S(1976^{1976}))) \equiv 7 \pmod{9}$, so $S(S(S(1976^{1976}))) = 7$.

Question 5: See Junior question 5.