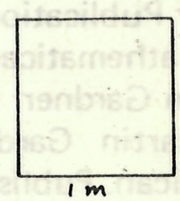
PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions, and the names of those who submit solutions before the publication of the next issue, will be published in Vol. 12 No. 3.

Although the problems are unclassified, the first 4 problems should require no more mathematical knowledge than is learnt in first or second form and the next 4 problems should require no more mathematical knowledge than is learnt in the first 4 forms of High School. Senior students are thus encouraged to attempt the later problems.

309. In a family with 6 children, the five elder children are respectively 2, 6, 8, 12 and 14 years older than the youngest. The age of each is a prime number of years. How old are they? Show that their ages will never again all be prime numbers (even if they live indefinitely).

310. A man had a square window with sides of length 1 metre as shown in the diagram. However, the window let in too much light and so he blocked up one half of it. How did he do this in such a way as to still have a square window which was 1 metre high and 1 metre wide?



311. In a classroom, there are 25 seats in a square array each occupied by a pupil. Each pupil moves to an adjacent seat to his right, left, front or rear, or stays in his seat. Prove that at least one pupil must in fact have stayed in his seat.

312. Suppose that a_1, a_2, a_3, a_4, a_5, a_6, a_7 are any seven integers and that b_1, b_2, b_3, b_4, b_5, b_6, b_7 are the same integers re-arranged. Show that the integer $(a_1-b_1)(a_2-b_2)(a_3-b_3)(a_4-b_4)(a_5-b_5)(a_6-b_6)(a_7-b_7)$ is even.

313. The King's men have captured a band of outlaws with an odd number of men. The rangers demand to know which ones shot the King's deer. The outlaws in panic each point to the nearest man. Prove that at least one man will not be accused. (Assume that no two pairs of outlaws are the same distance apart.)

314. Bob set himself the task of arranging all the positive rational numbers in a list. He did it as follows:

$a_1 = 1/1,\ a_2 = 1/2,\ a_3 = 2/1,\ a_4 = 1/3,\ a_5 = 2/2,\ a_6 = 3/1,\ a_7 = 1/4,\ a_8 = 2/3,$
$a_9 = 3/2,\ a_{10} = 4/1,\ a_{11} = 1/5, \ldots$

(Thus the rational number p/q precedes h/k in the list if $p+q < h+k$ or if $p+q = h+k$ and $p < h$.) His friend Joe asked how did he know that every rational number would appear in the list. Bob answered by writing down a formula giving the value of n when the rational number $p/q = a_n$ would appear. Joe, still unconvinced, wanted to know what the 1001'st number in the list would be. After a few calculations Bob answered him. Duplicate Bob's formula and find a_{1001}.

315. A large supply of small tiles is available for tiling the flat bottom of a large swimming pool. Each tile is in the shape of a regular polygon with edges all 1 cm long, and exactly 3 different shapes are used. The tiles are laid edge to edge in such a way that, although the vertices of 3 different tiles sometimes come together at the same point, no more than 3 vertices ever come together at the same point. Whenever 3 vertices do come together, the tiles at that point have different shapes. Prove that no tile used has an odd number of edges.

316. The rational numbers 169/30 and 13/15 are such that their sum is the same as their quotient: $(169/30) + (13/15) = 13/2 = (169/30)/(13/5)$. Find all pairs of rational numbers which have this property.

317. Let a and b be positive integers and define $a_1 = \sqrt{(ab)}$, $b_1 = \frac{1}{2}(a+b)$, $a_2 = \sqrt{(a_1 b_1)}$, $b_2 = \frac{1}{2}(a_1+b_1)$, ... Thus, in general, $a_{n+1} = \sqrt{(a_n b_n)}$, $b_{n+1} = \frac{1}{2}(a_n+b_n)$. Show that

$$|b_n - a_n| \leqslant |b-a|/2^n$$

for each positive integer n.

318. Show how to place squares with sides of length $(1/m)$ cm, where $m = 2, 3, 4, 5, \ldots$ (an infinite number of them) inside a square with side of length 1 cm. None of the squares you use is allowed to overlap any other one.

319. A rectangular box has sides of length x cm, y cm and z cm where x,y,z are different numbers. The perimeter of the box is $p = 4(x+y+z)$, its surface area is $s = 2(xy+yz+zx)$ and the length of its main diagonal is $d = \sqrt{(x^2 + y^2 + z^2)}$. Show that the length of the shortest side is less than $[\frac{1}{4}p - \sqrt{(d^2 - \frac{1}{2}s)}]/3$ and the length of the longest side is greater than $[\frac{1}{4}p + \sqrt{(d^2 - \frac{1}{2}s)}]/3$.

320. A large square is divided into one small square (with side of length s cm) and four rectangles A, B, C and D which are not squares. No side of any rectangle is the same length as a side of another nor the side of the big square. The sides of A are 4s cm and 2s cm. B has the largest area of any of the rectangles. C has sides in the ratio 3:1 and its area is 300 sq. cm. Find the area of D.

Solutions to Problems 297–308 (Vol. 12 No. 1)

297. A man has 3 bottles which hold exactly 8 litres, 5 litres and 3 litres. The two smaller bottles are empty but the largest one is full of wine which the man wishes to share with a friend. Without using any other means of measurement or any other container, how can he divide the wine into two equal amounts of 4 litres each?