318. Show how to place squares with sides of length $(1/m)$ cm, where $m = 2, 3, 4, 5, \ldots$ (an infinite number of them) inside a square with side of length 1 cm. None of the squares you use is allowed to overlap any other one.

319. A rectangular box has sides of length x cm, y cm and z cm where x,y,z are different numbers. The perimeter of the box is $p = 4(x+y+z)$, its surface area is $s = 2(xy+yz+zx)$ and the length of its main diagonal is $d = \sqrt{(x^2 + y^2 + z^2)}$. Show that the length of the shortest side is less than $[\frac{1}{4}p - \sqrt{(d^2 - \frac{1}{2}s)}]/3$ and the length of the longest side is greater than $[\frac{1}{4}p + \sqrt{(d^2 - \frac{1}{2}s)}]/3$.

320. A large square is divided into one small square (with side of length s cm) and four rectangles A, B, C and D which are not squares. No side of any rectangle is the same length as a side of another nor the side of the big square. The sides of A are 4s cm and 2s cm. B has the largest area of any of the rectangles. C has sides in the ratio 3:1 and its area is 300 sq. cm. Find the area of D.

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297. A man has 3 bottles which hold exactly 8 litres, 5 litres and 3 litres. The two smaller bottles are empty but the largest one is full of wine which the man wishes to share with a friend. Without using any other means of measurement or any other container, how can he divide the wine into two equal amounts of 4 litres each?

Answer: The quickest way is as follows:

(i)	Fill the 5 litre bottle from the 8 litre bottle	(0,5,3)
(ii)	Fill the 3 litre bottle from the 5 litre bottle	(3,2,3)
(iii)	Transfer the 3 litres from the 3 litre bottle to the 8 litre bottle	(0,2,6)
(iv)	Transfer the 2 litres from the 5 litre bottle to the 3 litre bottle	(2,0,6)
(v)	Fill the 5 litre bottle from the 6 litres in the 8 litre bottle	(2,5,1)
(vi)	Fill the 3 litre bottle with 1 litre from the 5 litre bottle	(3,4,1)
(vii)	Repeat (iii)	(0,4,4)

The figures at the end of each operation represent the number of litres in the 3 litre, 5 litre and 8 litre bottles respectively after that operation. (See also the article on page 2.)

298. $1^3 + 2^3 + 3^3 = 36$, which is divisible by 18. Find all sets of three consecutive natural numbers such that the sum of their cubes is divisible by 18.

Answer: From the identity $(n-1)^3 + n^3 + (n+1)^3 = 9[(n-1)n(n+1)/3 + n]$ we see that the sum of the three cubes is always divisible by 9. Note that $(n-1)n(n+1)/3$ is an integer since one of three consecutive integers is a multiple of 3. Indeed this term is an even integer since at least one of 3 consecutive integers is even. Hence 2 (and therefore 18) is a factor of $(n-1)^3 + n^3 + (n+1)^3$ if and only if n is even.

299. It is not hard to show that the trinomial $x^4 + px^2 + q$ is divisible by $x^2 + 1$ if and only if $p = q + 1$. For the general quadratic $x^2 + ax + b$, find the values of p, q such that $x^4 + px^2 + q$ is divisible by $x^2 + ax + b$.

Answer: Set $x^4 + px^2 + q = (x^2 + ax + b)(x^2 + cx + d)$

$$= x^4 + (a + c)x^3 + (b + ac + d)x^2 + (ad + bc)x + bd.$$

Equating the coefficients of x^3 and x on the R.H.S. to 0 gives $c = -a$, and $ad = -bc = ba$.

Hence if $a \neq 0$, $d = b$ and we obtain $p = b + ac + d = 2b - a^2$, $q = bd = b^2$. If $a = 0$, d is not determined; but we have $p = b + d$ and $q = bd$, which (on elimination of d) yield $q = pb - b^2$.

300. **Prove that if 1/(ab) + 1/(bc) + 1/(ca) = 1/(ab + bc + ca) then the sum of two of the numbers a, b and c is zero.**

Answer: Each of the following statements can be seen to be equivalent to the one above it, whilst the first statement is equivalent to the one in the question (above):

$$c/(abc) + a/(abc) + b/(abc) = 1/(ab + bc + ca)$$
$$(a + b + c)(ab + bc + ca) = abc$$
$$(a + b)(ab + bc + ca) + c(ab + bc + ca) - abc = 0$$
$$(a + b)(ab + bc + ca + c^2) = 0$$
$$(a + b)[b(a + c) + c(a + c)] = 0$$
$$(a + b)(a + c)(b + c) = 0$$
$$(a + b) = 0 \text{ or } (a + c) = 0 \text{ or } (b + c) = 0.$$

301. By moving the digits in the following magic square (and using no other operation), find 9 numbers in 3 rows and 3 columns such that when the numbers in any row or column or diagonal are *multiplied* together, you get the same answer. For example, the top row might be replaced by 7, 202 and 52. Magic squares are discussed in the article on page 2.

27	20	25
22	24	26
23	28	21

Answer: A diabolical trick! Move the second digit in each number up until it becomes an exponent.

2^7	2^0	2^5
2^2	2^4	2^6
2^3	2^8	2^1

302. A, B, C and D, are four points, in that order, on a straight line.

(i) If AB* = CD*, show that for any point P in the plane, PA* + PD* $\geqslant$ PB* + PC*.

(ii) Conversely, if PA* + PD* $\geqslant$ PB* + PC* for every position of P, show that AB* = CD*.

Answer:

(i)

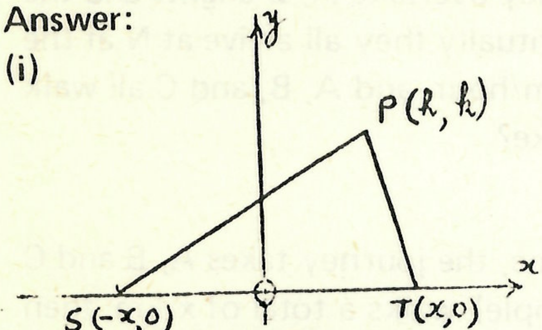


Figure 1

Let D(x) be the sum of the distances PS* + PT* in Fig. 1., where h,k > 0.

$D(x) = \sqrt{((h-x)^2 + k^2)} + \sqrt{((h+x)^2 + k^2)}$.

We show that D(x) increases as x increases by showing that D′(x) is always positive.

$D'(x) = -(h-x)/\sqrt{((h-x)^2 + k^2)} +$
$+ (h+x)/\sqrt{((h+x)^2 + k^2)}$

If $x \geqslant h$, neither term is negative and we are finished.

If $x < h$, $D'(x) = 1/X - 1/Y$

where $X = \sqrt{(1 + k^2/(h+x)^2)}$

$Y = \sqrt{(1 + k^2)/(h-x)^2)}$

Since $Y > X$, $1/X > 1/Y$ and so again $D'(x) > 0$.

Now, in Fig. 2., where 0 is the mid point of BC (and so of AD):

PB* + PC* = D(b) where b = OB*

and PA* + PD* = D(b + d) where d = AB*

Since D′(x) is always positive

D(b + d) > D(b).

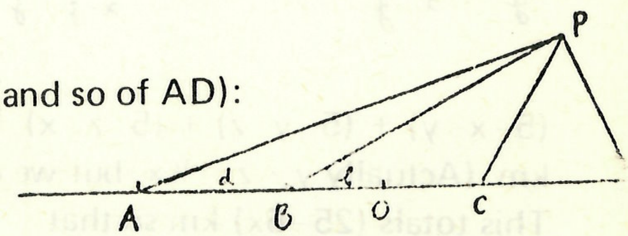


Figure 2

If P is on the line ($k = 0$), re-examination of the argument shows that $D'(x) = 0$ for $x \leq h$, whence $D(b) = D(b + d)$ provided $h \geq b + d$. This is also quite obvious directly since, in Fig. 3.,

$$\begin{aligned} PA^* + PD^* &= (PB^* + BA^*) + PD^* \\ &= PB^* + (DC^* + PD^*) \\ &= PB^* + PC^*. \end{aligned}$$

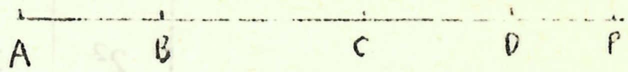
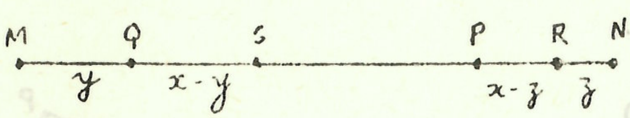


Figure 3

(ii) Since the given statement is true for every position of P, it is true when P is at A, i.e. $AD^* \geq AB^* + AC^*$. But it is obvious from Fig. 3, that $AD^* = AC^* + CD^*$ and so $CD^* \geq AB^*$. Similarly, by taking P to be at D, we can show that $AB^* \leq CD^*$ and so $AB^* = CD^*$.

303. Four men A, B, C and D set out simultaneously from M to reach N, 5 kilometres away. One of them, D, owns a motorcycle. He gives A a lift for part of the way, then turns back and picks up B. When they overtake A, B alights and the unselfish D once more turns back to assist C. Eventually they all arrive at N at the same moment. If D always travels at a steady v km/hour, and A, B, and C all walk at w km/hr, how long did the trip from M to N take?

Answer: Since they all arrive at N at the same time, the journey takes A, B and C exactly the same time (say t hrs). If A (for example) walks a total of x km, then he rides $(5-x)$ km on the motor cycle and so his time for the journey is $x/w + (5-x)/v = t$ hrs. Thus $x = (tvw-5w)/(v-w)$, and similarly B and C both walk for a total of x km where x has this value. Now suppose that D gives A a lift to the point P, picks up B at Q, overtakes A at R and then picks up C at S, where $MQ^* = y$ km, $RN^* = z$ km. Then the total distance travelled by D is $MP^* + PQ^* + QR^* + RS^* + SN^* = (5-x) + (5-x-y) + (5-y-z) + (5-z-x) + (5-x)$ where $y + z$ = distance walked by B = x km. (Actually $y = z = \frac{1}{2}x$, but we can manage without proving this.)



This totals $(25-6x)$ km so that

$vt = 25-6x = (25v + 5w - 6tvw)/(v-w)$, giving

$$t = (25v + 5w)/v(v + 5w).$$

304. Prove that a ray of light, having been reflected from three mutually perpendicular mirrors in turn becomes parallel to its original direction but in the opposite sense.

Answer: To do this question we make use of ordered triples (x,y,z) of real numbers to represent points in 3 dimensional space, just as ordered pairs are used for the number plane.

In Fig. 1 AB represents a ray of light reflected along BC from a plane mirror. CB is produced back to A′ where BA′* = BA* and the laws of reflection assure us that the plane of ABA′ is perpendicular to the mirror (whence the perpendicular AM from A to the mirror lies in this plane) and that ∠MBA = ∠MBA′. It is now easy to prove the triangles ABM and A′BM congruent, so AM* = A′M*.

In Fig. 2, we have chosen axes so that x 0 y is parallel to the mirror, and PO is parallel to the ray AB. If P is the point (h,k,l) we see that the reflected ray is parallel to OQ or to P′O where P′ is the point obtained by dropping a perpendicular to x 0 y and doubling it. Hence the reflected ray is parallel to a line from $(h,k,-l)$ towards 0. Similarly a ray parallel to PO would be reflected from a mirror parallel to x 0 z to the direction from $(h,-k,l)$ towards 0, and from a mirror parallel to y 0 z in the direction from $(-h,k,l)$ towards 0.

Now if a ray parallel to the direction from (h,k,l) towards 0 is reflected from three mirrors one parallel to each co-ordinate plane, the final result will be a ray travelling in the direction from $(-h,-k,-l)$ towards 0, i.e. in the opposite direction to the incident ray.

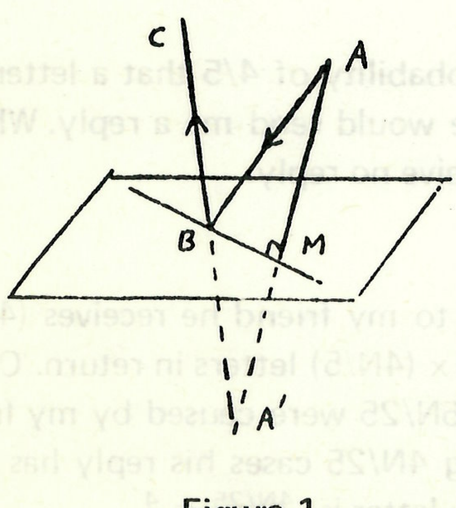


Figure 1

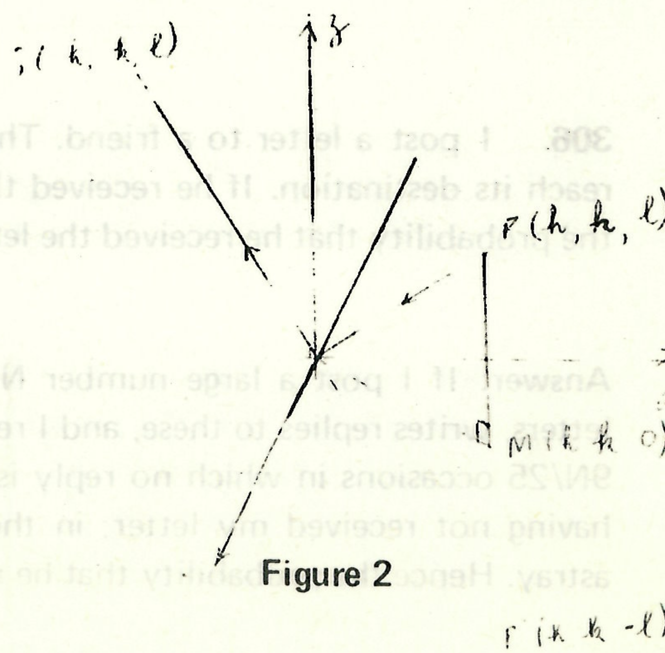


Figure 2

305. An aeroplane leaves a town of latitude 1°S, flies x km due South, then x km due East, then x km due North. He is then 3x km due East of his starting point. Find x.

Answer: The question should have read l°S rather than 1°S. If the plane starts at point A, flies south to B, east to C and then north to D, the difference in longitude between A and D is equal to that between B and C. Since the arc AD of one small circle (parallel of latitude) is three times the arc BC of the other, it follows that the radius of the first small circle is also three times that of the second.

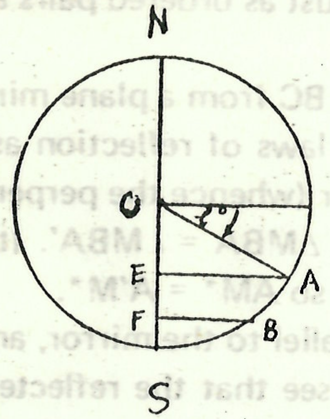


Figure 1

Fig. 1 represents the great circle (meridian of longitude) through A and B. The circle of latitude through A has radius AE* = R cos l° (where R = radius of the earth). Hence BF* = AE*/3 = (R cos l°)/3 = R cos θ° where θ° is the lattiude of B. Solving, cos θ° = (cos l°)/3. Hence the angle $\angle$BOA = $\theta° - l°$ and X = R.($\angle$BOA) *where $\angle$BOA is expressed in radian measure.*
i.e. $X = \pi R(\theta - l)/180$ where $\cos\theta° = (\cos l°)/3$.
Taking $l = 1$, we get $\theta° \approx 70°32'$, $\pi\theta/180 \approx 1.231$ radians and $\pi l/180 \approx .017$ radians, and taking the radius of the earth to be 6,380 km we obtain X ≈ 7,740 km.

306. I post a letter to a friend. There is a probability of 4/5 that a letter will reach its destination. If he received the letter he would send me a reply. What is the probability that he received the letter if I receive no reply.

Answer: If I post a large number N of letters to my friend he receives (4/5)N letters, writes replies to these, and I receive (4/5) x (4N/5) letters in return. Of the 9N/25 occasions in which no reply is received, 5N/25 were caused by my friend having not received my letter; in the remaining 4N/25 cases his reply has gone astray. Hence the probability that he received my letter is $\frac{4N/25}{9N/25} = \frac{4}{9}$.

307. I have 5 balls, identical in appearance, of which two are unequal in weight, one heavier and one lighter than each of the other 3. Together these 2 are equal in weight to two regular balls. Show how to distinguish the balls in three comparisons using a beam balance.

Answer: Label the balls A, B, C, D and E. First weigh A against B, then C against D. On at least one occasion, and perhaps on both, a balance was not obtained.

Case 1. One pair balanced. We may assume (perhaps after relabelling the balls) that A balanced B and C was heavier than D. Now weigh A against E. Since A is now known to be regular this identifies E. If E is regular, C is heavy and D is light. If E is heavy, C is regular and D is light. If E is light, C is heavy and D is regular.

Case 2. Neither pair balances. We may suppose that A is heavier than B, C heavier than D. Weigh A against the regular ball E. If A and E balance then A is regular, B is light and C is heavy. If A is heavy, D is the light ball.

308. Seven towns $T_1, T_2, \ldots T_7$ are connected by a network of 21 one-way roads such that exactly one road runs directly between any 2 towns. (For example, the towns could be situated at the vertices of a convex heptagon, the seven sides and 14 diagonals of which form the network of roads). Given any pair of towns T_i, T_j $(1 \leqslant i < j \leqslant 7)$ there is a third town, T_k, such that T_k can be reached by a direct route from both T_i and T_j.

(i) Prove that of the 6 roads with an end at any town T_i, the number in which traffic is directed away from T_i is at least 3. Hence prove that it is exactly 3.

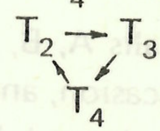
(ii) Let the towns which can be reached directly from T_1 be numbered T_2, T_3, T_4. Show that the roads between T_2, T_3, T_4 form a circuit.

(iii) Display on a sketch a possible orientation of traffic on the 21 roads.

Answer: Let us write $T_i \to T_k$ to denote that T_k can be reached by a direct route from T_i. If $i \neq j$, let us write $T_i + T_j$ for the town T_k with the smallest value for k such that $T_i \to T_k$, $T_j \to T_k$. Clearly, $k \neq i$ or j.

(i) For simplicity, take i = 1 and suppose $T_1 \to T_2$ and $T_1 + T_2 = T_3$. $T_1 + T_3 \neq T_2$ since there is only one road between T_2 and T_3, and $T_2 \to T_3$. Thus there are at least three roads $T_1 \to T_2$, $T_1 \to T_3$, $T_1 \to T_1 + T_3$ from T_1, and similarly from each T_i we have already accounted for all 21 roads in the network, and it is impossible that a fourth road should leave any T_i.

(ii) Using the same notation as above, we have $T_1 + T_3 = T_4$. Since there are only 3 towns which can be reached directly from T_1, $T_1 + T_4$ must be T_2 or T_3. Since $T_3 \to T_4$, $T_1 + T_4 \neq T_3$ and so $T_1 + T_4 = T_2$, i.e.



(iii)