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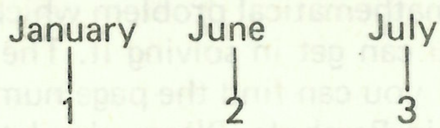
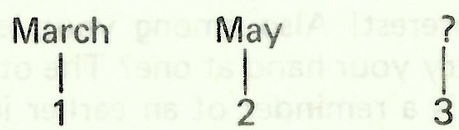
COUNTING LARGE SETS

Everyone knows how to count the number of elements of a finite set. But what are we actually doing when we count? For example we know that the set J = {January, June, July} and W = {1,2,3} have the same number of elements, but the sets M = {March,May} and W = {1,2,3} have different numbers of elements. Why?

To answer this question, write down the elements of J and then write down the elements of W below them. The following diagram shows one way in which we can join the elements of J to the elements of W by lines.



In this case, each element of both sets has one line, and only one line, drawn to it. However, if we try this for the sets M and W, we can easily see that it is impossible:



We will say that two sets are in *one-to-one correspondence* if we can draw such lines satisfying the following condition:

> Each element of either set is joined to one, and only one, element of the other set.

Thus the sets J and W are in one-to-one correspondence but the sets M and W (or the sets J and M) are not. You might now like to show that the set of days in a week is in one-to-one correspondence with the set {1,2,3,4,5,6,7} and the set of letters in our alphabet is in one-to-one correspondence with the set {1,2,3, . . . ,26}.

We are now in a position to "count" finite sets. We will say that the *cardinal* of a set is 1 if it is in one-to-one correspondence with the set {1}, the cardinal of a set is 2 if it is in one-to-one correspondence with the set {1,2}, and so on. For example, the cardinal of our set J is 3, the cardinal of our set M is 2 and the cardinal of the set of all letters in our alphabet is 26.

In the case of finite sets, our word "cardinal" seems unnecessary since it just means the number of elements in the set. However, we do get a bonus from the word when we use it for infinite sets. An aid to thinking about infinite sets is a story told by David Hilbert, a mathematician who lived in Germany at the beginning of this century. There were once two hotels in which each room would take only one person. The owners knew that their hotels were full when the set of guests was in one-to-one correspondence with the set of rooms:

room 1	room 2	room 3	. . . . .	room "n"	. . .
|	|	|		|	
person in room 1	person in room 2	person in room 3	. . . . .	person in room "n"	. . .

One day when both hotels were full, a traveller arrived at the first hotel (which was called "Hotel Finite") and the owner regretfully had to send him away. So the traveller went to "Hotel Infinite" (the other hotel). When he asked for a room, the owner replied that, although his hotel was full, he could still accommodate the traveller because his hotel had an infinite number of rooms. He simply transferred the occupant of room 1 to room 2, the one from room 2 to room 3, and so on, and the new arrival was then able to have room 1. This illustrates the remarkable fact that an infinite set can be in one-to-one correspondence with a subset, in the case of this story the one-to-one correspondence being between the room out of which a guest was moved and the room into which he was moved:

room 1	room 2	room 3	. . .	room n	. . .
|	|	|		|	
room 2	room 3	room 4	. . .	room n+1	. . .

In fact, last century, Dedekind noted that a set is infinite if and only if it is in one-to-one correspondence with a subset (besides itself of course!).

We are now in a position to "count" infinite sets – or rather to assign cardinals to infinite sets. For example, what is the cardinal of the set N = {1,2,3, . . .} of all positive integers? Clearly it is not 1 since N and {1} are not in one-to-one correspondence. Similarly, the cardinal of N is not 2, 3 etc. In fact if n is a positive integer and the cardinal of N were n, then we would have a one-to-one correspondence:

1	2	3	. . .	n	?	
|	|	|		|	|	
1	2	3	. . .	n	n+1	. . .

So N is infinite and we have to invent a new symbol $\aleph_0$ for the cardinal of any set in one-to-one correspondence with N. ($\aleph$ is the first letter of the Hebrew alphabet and is called "aleph" – $\aleph_0$ is called "aleph-nought").

Now let us find the cardinals of some more infinite sets:

(1) The set {2,4,6,8, . . .} of all positive even integers. At first sight, you might think that the cardinal of this set was smaller than $\aleph_0$, but we can show that this

set is actually in one-to-one correspondence with N:

$$\begin{array}{ccccccc} 2 & 4 & 6 & 8 & \dots & 2n & \dots \\ | & | & | & | & & | & \\ 1 & 2 & 3 & 4 & \dots & n & \dots \end{array}$$

Thus by our rule the cardinal of this set is $\aleph_0$. You might now like to take some other infinite subset of N, such as the set of odd positive integers or the set of perfect squares and show that the cardinal of these sets is again $\aleph_0$ (Galileo did this for the set of perfect squares 300 years ago.)

(2) The set {0,1,2, . . .} of all positive integers together with 0. As before, you might think that the cardinal of this set was bigger than $\aleph_0$ but, using Hilbert's infinite hotel trick, we can show that this set is in one-to-one correspondence with N and so its cardinal is also $\aleph_0$.

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & n-1 & \dots \\ | & | & | & | & & | & \\ 1 & 2 & 3 & 4 & \dots & n & \dots \end{array}$$

Similarly, you might like to show that the cardinal of the sets {-1,0,1,2, . . .}, {-2,-1,0,1, . . .} etc, is $\aleph_0$.

(3) The set J of all integers. We surely have a bigger cardinal now because, just as there is no number bigger than all the positive integers, so there is no number smaller than all the negative integers and so we have nowhere to start. But here we can use a trick of *re-arranging* the integers by alternately writing down the positive and negative numbers. The result looks like this:

$$\begin{array}{cccccccccc} 0 & 1 & -1 & 2 & -2 & 3 & -3 & \dots & n & -n & \dots \\ | & | & | & | & | & | & | & & | & | & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots & 2n & 2n+1 & \dots \end{array}$$

Once again, we have our familiar one-to-one correspondence between our set J and N, and so the cardinal of J is $\aleph_0$.

(4) The set Q^+ of all positive rational numbers. Surely the cardinal of this set is bigger than $\aleph_0$ as it is so huge! But this is not the case as is shown in problem 314 where a one-to-one correspondence between Q^+ and N is given. To do this, we write the positive rationals down as follows

$$\begin{array}{ccccc} 1/1 & 1/2 & 1/3 & 1/4 & \dots \\ 2/1 & 2/2 & 2/3 & \dots & \\ 3/1 & 3/2 & 3/3 & \dots & \\ 4/1 & 4/2 & \dots & & \end{array}$$

and "count" them diagonal by diagonal (following the arrows). The one-to-one correspondence given in problem 314 is:

1/1	1/2	2/1	1/3	2/2	. . .	p/q	. . .
|	|	|	|	|		|	
1	2	3	4	5	. . .	$\frac{1}{2}(p+q-1)(p+q-2)+p$	. . .

Certainly some rational numbers occur more than once (such as 1/1 = 2/2 = 3/3 = . . .) but this can be overcome by leaving that number out after its first occurrence and moving the remainder one place to the left each time we do this. Now a trick similar to example 3 will show us that the cardinal of the set of all rational numbers is also $\aleph_0$.

By now we are ready to believe that the cardinal of any infinite set is $\aleph_0$. However in 1874, George Cantor – who was the first mathematician to use the idea of cardinals – succeeded in proving that the cardinal of the set R of all real numbers was bigger than $\aleph_0$.

This now raises a question which has puzzled mathematicians for years: are there any sets whose cardinal is bigger than $\aleph_0$ but smaller than the cardinal of R? The belief that there are no such sets is called the "continuum hypothesis" and, for many years, no progress was made in proving it or disproving it. However, in 1963 P.J. Cohen showed in effect that it could neither be proved nor disproved (in the same way as you can "prove" that you cannot prove that there is only one line passing through some point and parallel to some line). Cantor *was* able to prove that there were cardinals bigger than the cardinal of R (in fact an infinite number of them).

Questions:

(1) Find a subset of {1,2,3} which is in one-to-one correspondence with the set {March,May}. Can you find a subset of {March,May} which is in one-to-one correspondence with the set {1,2,3}?

(2) Write down *any* two *finite* sets. Show that the smaller of the two sets is in one-to-one correspondence with a subset of the larger set. Show that the larger of the two sets is not in one-to-one correspondence with a subset of the smaller set.

(3) Use question 2 to invent a way of deciding whether one cardinal is bigger than another.

(4) By experimenting with ordered pairs (a,b) of elements $a \in A$ and $b \in B$ for finite sets A,B, invent a way of multiplying two cardinals.