DRUNKEN ELECTRONS WHICH GAMBLE, FACE RUIN

What is similar between a drunk staggering along a narrow alley and an electron moving along a wire under a potential difference? Well, to a reasonable degree of accuracy the two situations can be modelled by the same mathematical setup -- called a "random walk".

We will restrict our attention to a walk in one dimension with the step-size being unity, we presume there is a probability p of moving in one direction along the axis (alley or wire) and consequently a probability $q = 1-p$ of moving in the other direction. To get a very simple model, we will assume each movement occurs after a time interval of unity also. Now all this may seem fairly restrictive, but the model is still of wide applicability; for example, we have also modelled a gambling game. Let us continue the discussion in terms of a series of gambling games, and let us investigate the probability of the gambler's ruin.

The total capital of *a* units is initially split between the gambler (let's call him Zed) and the bank of the house, so that Zed may have z units and the bank $a-z$ units of cash. Zed plays the house a series of games, until the bank loses all its cash, or Zed loses all his (the gambler's ruin). Our unit of "time" is each game. In Fig. 1, the horizontal axis is "time" (the number of games) and the vertical axis Zed's cash (starting with say $z = 12$) for the case $a = 20$, and $p = 0.4$ ($q = 0.6$).

We are interested in the *probability* that Zed will ultimately be ruined, when he has cash z to start with. Let this probability be represented by $Q(z)$, and suppose Zed's cash increases to $z+1$ with probability p and decreases to $z-1$ with probability q in one game. Since Zed will either win or lose the next game,

$$Q(z) = pQ(z+1) + qQ(z-1) \qquad \text{(i)}$$

This is a second-order difference equation for $Q(z)$, and can be solved by assuming

$$Q(z) = m^z \text{ for some constant } m \qquad \text{(ii)}$$

Substituting (ii) into (i) and dividing by m^{z-1}

$$m = pm^2 + q \text{ whence}$$
$$m = [1+\sqrt{(1-4pq)}]/(2p) \text{ or } m = [1-\sqrt{(1-4pq)}]/(2p)$$

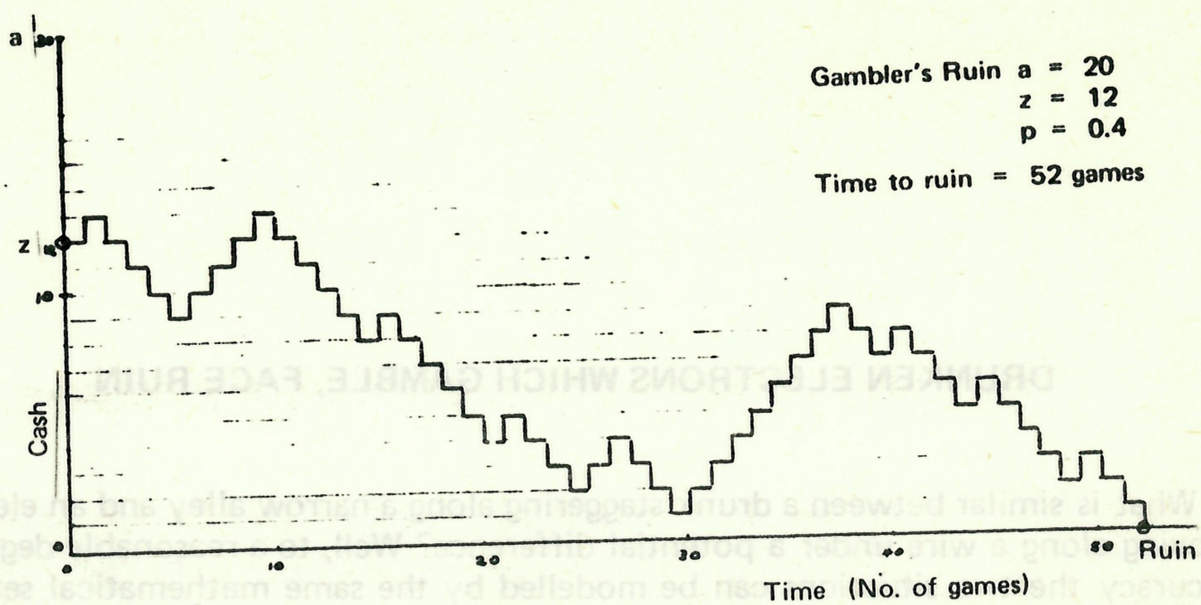

Figure 1

Either root satisfies, and so will the most general combination

$$Q(z) = m^z = Am_1{}^z + Bm_2{}^z$$

where $m_1 = (1 + \sqrt{[1-4p(1-p)]})/(2p)$ (remembering that $q = 1-p$)

$$= [1 + \sqrt{(2p-1)^2}]/(2p) = 1$$

and similarly, for the negative square root, $m_2 = q/p$, which we will write as r.

So $$Q(z) = A + Br^z \qquad \text{(iii)}$$

But we know that, if $z = 0$, the probability of ultimate ruin is 1 (certainty), while of $z = a$, the probability of ultimate ruin is 0. i.e.

$$Q(0) = 1 = A + B \qquad \text{(iv)}$$

$$Q(a) = 0 = A + Br^a \qquad \text{(v)}$$

and, solving for A and B yields

$$Q(z) = (r^a - r^z)/(r^a - 1) \qquad \text{(vi)}$$

Check that $Q(0) = 1$ and $Q(a) = 0$.

In the case illustrated in Fig. 1, $p = 0.4$, $q = 0.6$ so $r = 1.5$ and, for $z = 12$, $a = 20$ and $Q(12) = 0.99$. Not very hopeful.

Suppose we double the stakes in the game. We don't have to go through all the working again, since doubling the stakes is equivalent to halving the initial cash, so, in this new game ($r = 1.5$ still) $Q(12) = (r^{10} - r^6)/(r^{10} - 1) = 0.81$.

The improved situation is not surprising because the larger stakes make the process more erratic – imagine magnifying the walk in Fig. 1 by a factor of two, but the boundaries remain unchanged (i.e. the steps are magnified); because $q > p$, the long range trend is downwards, but with a more erratic process, the *chance* of reaching *a* before the downward drift is increased.

What happens to Q(z) if the house is infinitely rich (poor Zed!). We can let $a \to \infty$ in (vi) and so

$$Q(z) \to 1 \quad \text{if } q > p \ (r > 1)$$
$$\to r^z \quad \text{if } q < p \ (r < 1)$$

Zed's hopes may be high but unless $p > q$ he will ultimately be ruined, and even if $p > q$ his prospects are not wonderful. We can be pretty sure any casino which has been operating for any reasonable length of time does not have games in which $p > q$ (although p will not be much smaller than 0.5, or people would give up playing). So, if the stakes are relatively high compared with the cash needed, Zed might be prepared to take the risk. Over many games the average gain can be defined as the sum of two terms – the amount to be won multiplied by the probability of winning, less the amount to be lost by the probability of losing. Technically, we define

Expected gain = Gain x Probability of winning
− Loss x Probability of Losing

So for Zed's expected gain we have

Expected gain = E(G) (giving it a symbol)

$$= (a-z)\,P(z) - zQ(z)$$
$$= (a-z)(1-Q(z)) - zQ(z)$$
$$= (a-z) - aQ(z)$$
$$= a(1-Q(z)) - z$$
$$= a\,P(z) - z$$

By using $P(z) = 1-Q(Z)$, the probability of ultimately winning, we are presuming Zed is going to play to the bitter (or sweet) finish. If he adopts a policy of stopping play before the end under certain eventualities, the probabilities are changed, of course. The importance of expected gain is that the gain need not be purely financial, but can involve all that the "player" has to gain. This is why life assurance, for example, is an acceptable (if usually unprofitable) risk, because the gains in peace of mind, etc, are of value, and the losses, if they chance to occur are so severe.

If Zed plays out the game series, how many games are to be expected – we talk of the expected duration of the series. Let the expected duration (when Zed has capital z) be D(z). If Zed wins the first game, he has now cash $z + 1$, and the expected duration is $1 + D(z+1)$, the 1 coming from the one game already played. So, as with equation (i) at the beginning, it follows

$$D(z) = p\,D(z+1) + q\,D(z-1) + 1 \qquad \text{(vii)}$$

Notice that this is very similar to equation (i), and can be solved in a very similar manner, but the boundary conditions are different, since

$$D(0) = 0$$
$$D(a) = 0 \qquad \text{(viii)}$$

Let's try to fix up the 1 first. Because $(p+q)z = z$, we might try as a solution to eliminate the 1, the following:

$$D^*(z) = kz \tag{ix}$$

which, when substituted, gives

$$kz = pk(z+1) + qk(z-1) + 1 = (p+q)kz + pk - qk + 1 \text{ and so}$$
$$(q-p)k = 1, \text{ or } k = 1/(q-p). \text{ Thus}$$
$$D^*(z) = z/(q-p) \tag{x}$$

If we set $D(z) = D^*(z) + D_*(z)$, substituting in (vii) leaves us to find $D_*(z)$ satisfying

$$D_*(z) = pD_*(z+1) + q\,D_*(z-1) \text{ exactly as in (i).}$$

As we found there,

$$D_*(z) = A + Br^z \quad \text{where } r = q/p$$

and

$$D(z) = A + Br^z + z/(q-p)$$

Applying conditions (viii) shows, by solving two equations for A and B,

$$D(z) = \frac{z}{q-p} - \frac{a(1-r^z)}{(q-p)(1-r^a)} \tag{xi}$$

As an example, if Zed has \$90 and the total cash is \$100, with $q = 0.6$, $(p = 0.4)$, with stake \$1, then by (vi), $Q(90) = 0.983$ and by (xi), $D(90) = 441.3$.

If Zed plays out the game till the bitter/sweet end his expected gain, from the definition, is

$$\begin{aligned} E(G) &= P(z) \,.\, (a-z) - Q(z) \,.\, z \\ &= 0.017 \times \$10 - 0.983 \times \$90 \\ &= -\$88.30 \end{aligned}$$

I hope Zed is gambling just for the thrill he gets out of it, for he is not going to get much financially out of it, almost surely. But, there is always that way-out chance . . . !

Returning to an electron. We might model the movement of an electron along a wire by having it "jump" from one atom to the next every time unit (possibly of the order of 10^{-16} sec.!). If there is no applied voltage, under thermal agitation, we might assume the electron jumps either to the right or to the left with equal probability, whereas, if the right-hand end of the wire is maintained at positive voltage, the electron might be assumed to jump to the right with a probability q and to the left with probability $p < q$. The model is simple, but instructive, and all the work done on the gambler's ruin is immediately applicable.

What we have shown is that we can build a reasonably simple mathematical model of a variety of different processes, and, solving the model equation involves the high school mathematics of solving a quadratic equation; while satisfying the initial conditions determines any constants in the model by solving two linear equations.

EXTENSIONS

1. If $p = q = \frac{1}{2}$, the solutions (vi) and (xi) break down. Show by substitution that for a "fair" game (define by $p = q = \frac{1}{2}$).

$$Q(z) = 1 - z/a$$

and

$$D(z) = z(a - z)$$

satisfy the equations (i) and (vii) as well as the boundary conditions ((iv) and (v), and (viii)).

2. Calculators might help you to check that when $p = 0.45$, $a = 10$, $z = 9$, then $Q(9) = 0.21$, $E(G) = -1.1$, and $D(9) = 11$.
3. What happens in a "fair" game against an infinitely rich house bank?
4. If the drunk mentioned in the opening paragraph still has 100 feet to get home, and every 10 sec. he takes a pace of size 2 feet but with a chance of one in five of going in the wrong direction, how long might he *expect* to take to get home? (Regard the road away from home as being infinitely long, and note that, as $a \to \infty$, $a/4^a \to 0$).

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[Mr Hannan is a tutor at the University of NSW in Statistics. This article is the talk he gave at this year's "4 unit day" organised by the Mathematics Association of NSW.]

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Large Number Plates

"My", said Professor A to Professor B as a large limousine sailed by, "that car has a large registration number." Sure enough, the vanishing number plate read

EXP 999.

"I wonder if there are any larger numbers around" mused Professor B.

After some thought, Professor B returned triumphant with the ultimate in large registration numbers:

TAN 90.

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