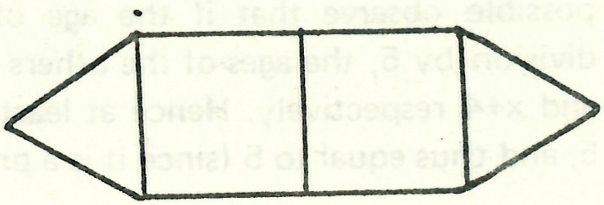
330. Certain convex polygons can be dissected into squares and equilateral triangles all having the same length of side. For example, the illustration shows a hexagon dissected in such a way. If a convex polygon can be dissected in this way, how many sides did it have originally? Prove your answer.



331. Suppose that n^2+1 boys are lined up shoulder-to-shoulder in a straight line. Show that it is always possible to select $n+1$ boys to take one pace forward so that going from left to right their heights are either increasing or decreasing.

332. In a number of years equal to the number of times a pig's mother is as old as the pig, the pig's father will be as many times as old as the pig as the pig is years old now. The pig's mother is twice as old as the pig will be when the pig's father is twice as old as the pig will be when the pig's mother is less by the difference in ages between the father and the mother than three times as old as the pig will be when the pig's father is one year less than twelve times as old as the pig is when the pig's mother is eight times the age of the pig.

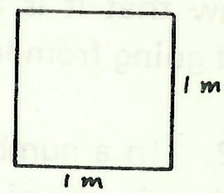
When the pig is as old as the pig's mother will be when the difference in ages between the pig's father and the pig is less than the age of the pig's mother by twice the difference in ages between the pig's father and the pig's mother, the pig's mother will be five times as old as the pig will be when the pig's father is one year more than ten times as old as the pig is when the pig is less by four years than one-seventh of the combined ages of his father and mother. FIND THEIR RESPECTIVE AGES. (For the purposes of this problem, the pig may be considered to be immortal.)

Solutions to Problems 309–320 (Vol. 12 No.2)

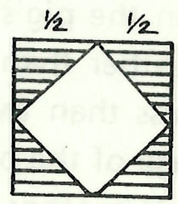
309. In a family with 6 children, the five elder children are respectively 2, 6, 8, 12 and 14 years older than the youngest. The age of each is a prime number of years. How old are they? Show that their ages will never again all be prime numbers (even if they live indefinitely).

Answer: Their ages are 5, 7, 11, 13, 17, 19. To prove that no other answer is possible observe that if the age of the youngest leaves the remainder x after division by 5, the ages of the others exceed a multiple of 5 by x+2, x+1, x+3, x+2 and x+4 respectively. Hence at least one of the ages must always be a multiple of 5, and thus equal to 5 (since it is a prime).

310. A man had a square window with sides of length 1 metre as shown in the diagram. However, the window let in too much light and so he blocked up one half of it. How did he do this in such a way as to still have a square window which was 1 metre high and 1 metre wide?



Answer:



311. In a classroom, there are 25 seats in a square array each occupied by a pupil. Each pupil moves to an adjacent seat to his right, left, front or rear, or stays in his seat. Prove that at least one pupil must in fact have stayed in his seat.

Answer: Imagine the seats to be coloured alternately black and white (like the squares of a chessboard) with the corner seats white. If everybody were to move, the 13 pupils occupying white seats would all move to a black seat, which (since there are only 12 black seats) is impossible.

312. Suppose that a_1, a_2, a_3, a_4, a_5, a_6, a_7 are any seven integers and that b_1, b_2, b_3, b_4, b_5, b_6, b_7 are the same integers re-arranged. Show that the integer $(a_1-b_1)(a_2-b_2)(a_3-b_3)(a_4-b_4)(a_5-b_5)(a_6-b_6)(a_7-b_7)$ is even.

Answer: Since the b's are just a re-arrangement of the a's,

$$(a_1-b_1)+(a_2-b_2)+(a_3-b_3)+(a_4-b_4)+(a_5-b_5)+(a_6-b_6)+(a_7-b_7)$$
$$=(a_1+a_2+a_3+a_4+a_5+a_6+a_7)-(b_1+b_2+b_3+b_4+b_5+b_6+b_7)=0$$

Now if all 7 factors of the above product were odd, then their sum would also be odd (since 7 is odd). Since their sum is 0, this is impossible and so at least one of the factors is even, making the product even.

313. The King's men have captured a band of outlaws with an odd number of men. The rangers demand to know which ones shot the King's deer. The outlaws in panic each point to the nearest man. Prove that at least one man will not be accused. (Assume that no two pairs of outlaws are the same distance apart.)

Answer: It is possible that some of the outlaws form pairs each of whom points at the other and such that neither is pointed at by anyone else. However, as the total number of outlaws is odd they cannot all be paired off in this fashion. Hence we must be able to find a "chain" {A,B,C,. . . ,K,L} of more than two outlaws, each outlaw pointing to the next one in the chain. Note that $*AB > *BC > *CD > \ldots > *KL$. If this chain is as long as possible, L must point to K. (If he pointed at a different outlaw, we have a longer chain; he cannot point to an earlier outlaw, say A, in the chain, since if $*LA < *LK < *AB$ then A could not have pointed to B.) There are now two easy ways of finishing off. The first member A of a chain of maximal length cannot be pointed at by anyone (since if so we would immediately have a longer chain). Alternatively, since at least two people accuse the 2nd last member of any chain of length greater than 2, and the total number of accusations is equal to the number of outlaws, at least one of these must go unaccused.

314. Bob set himself the task of arranging all the positive rational numbers in a list. He did it as follows:

$a_1 = 1/1,\ a_2 = 1/2,\ a_3 = 2/1,\ a_4 = 1/3,\ a_5 = 2/2,\ a_6 = 3/1,\ a_7 = 1/4,\ a_8 = 2/3,$
$a_9 = 3/2,\ a_{10} = 4/1,\ a_{11} = 1/5, \ldots$

(Thus the rational number p/q precedes h/k in the list if $p+q < h+k$ or if $p+q = h+k$ and $p < h$.) His friend Joe asked how did he know that every rational number would appear in the list. Bob answered by writing down a formula giving the value of n when the rational number $p/q = a_n$ would appear. Joe, still unconvinced, wanted to know what the 1001'st number in the list would be. After a few calculations Bob answered him. Duplicate Bob's formula and find a_{1001}.

Answer: Call h+k the "height" of the rational number h/k. (5/7 has height 12.) There are H−1 numbers of height H in the list viz. {1/(H−1), 2/H−2), . . . , (H−1)/1}. The total number of fractions of height ⩽ H is

$$1 + 2 + 3 + \ldots + (H-1) = \tfrac{1}{2}H(H-1).$$

(On the left hand side, 1 is the number of fractions of height two, 2 is the number of fractions of height 3, etc.)

The fraction p/q is preceded in the list by all the fractions of height < p+q−1, a total of ½(p+q−1)(p+q−2) fractions. It is also preceded (if p>1) by the p−1 fractions

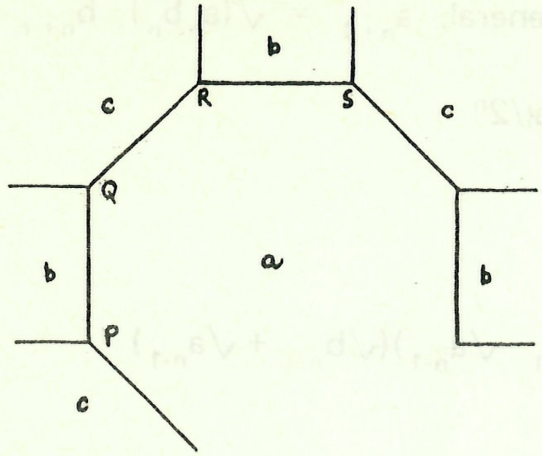
$$1/(p+q-1),\ 2/(p+q-2),\ \ldots,\ (p-1)/(q+1)$$

Hence $p/q = a_n$ where $n = \tfrac{1}{2}(p+q-1)(p+q-2) + (p-1) + 1$
$= \tfrac{1}{2}(p+q-1)(p+q-2) + p.$

To find a_{1001} observe that if H = 45, ½H(H−1) = 990
and if H = 46, ½H(H−1) = 1035. Hence p + q − 1 = 45
and p = 1001−990 = 11, so $a_{1001} = p/q = 11/35$

315. A large supply of small tiles is available for tiling the flat bottom of a large swimming pool. Each tile is in the shape of a regular polygon with edges all 1 cm long, and exactly 3 different shapes are used. The tiles are laid edge to edge in such a way that, although the vertices of 3 different tiles sometimes come together at the same point, no more than 3 vertices ever come together at the same point. Whenever 3 vertices do come together, the tiles at that point have different shapes. Prove that no tile used has an odd number of edges.

Answer: Let P be a point at which verticies of tiles of types a, b and c coincide. (See figure.)



By the given conditions the tile which meets a along QR must be of type c, then that which meets a along RS again of type b. In fact, tiles of types b and c must alternate strictly around the tile of type a, whence its number of edges is even. Similarly for tiles of the other types. [It is possible to tile subject to the conditions using tiles with 4, 6 and 12 sides, for example. Incidentally, it is quite clear that vertices of 3 tiles always come together. You might like to prove that the tiles with 4, 6 and 12 sides give the only possible answer.]

316. The rational numbers 169/30 and 13/15 are such that their sum is the same as their quotient: (169/30) + (13/15) = 13/2 = (169/30)/(13/5). Find all pairs of rational numbers which have this property.

Answer: Suppose the two numbers are x and y with $y \neq 0$. Then $x + y = x/y$, giving $xy + y^2 = x$ and so $x(1-y) = y^2$.

If $y = 1$, then $x + 1 = x$ (which is impossible) and so $x = y^2/(1-y)$.

Thus the pairs are $(y^2/(1-y), y)$ for any rational number y except 0 or 1.

Comment (Submitted by Mr J. Scott from Barker College)

Instead of solving for x, we could have solved for y and obtained

$$y^2 + xy - x = 0$$

Thus $y = \frac{1}{2}[-x \pm \sqrt{(x^2 + 4x)}]$, provided $x^2 + 4x \geqslant 0$.

There appear to be, therefore, *two* real solutions for y for each value of x with $x^2 + 4x \geqslant 0$. However, from the original solution, we have $x = y^2/(1-y)$ and so $x^2 + 4x = y^2(y-2)^2/(1-y)^2 \geqslant 0$. Thus the solutions for y become

$$y = [-y^2 \pm (y^2 - 2y)]/2(1-y)$$

yielding the two solutions $y = y$ and $y = -y/(1-y)$.

In the second case, $y = 0$ or 2. However $y \neq 0$ and so $y = 2$. Thus $x^2 + 4x = 0$ and so $y^2 + xy - x$ is a perfect square, giving only one value for y.

317. Let a and b be positive integers and define $a_1 = \sqrt{(ab)}$, $b_1 = ½(a+b)$, $a_2 = \sqrt{(a_1 b_1)}$, $b_2 = ½(a_1 + b_1)$, ... Thus, in general, $a_{n+1} = \sqrt{(a_n b_n)}$, $b_{n+1} = ½(a_n + b_n)$. Show that

$$|b_n - a_n| \leqslant |b-a|/2^n$$

for each positive integer n.

Answer: $b_n - a_n = ½(b_{n-1} + a_{n-1}) - \sqrt{(a_{n-1} b_{n-1})}$

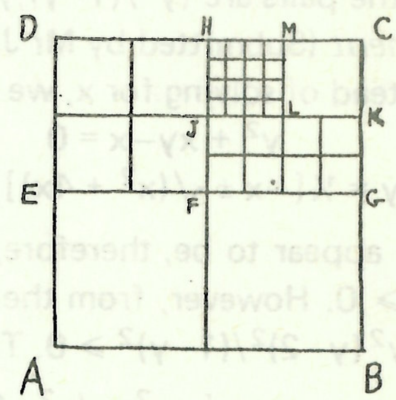
$$= ½(\sqrt{b_{n-1}} - \sqrt{a_{n-1}})^2 \leqslant ½(\sqrt{b_{n-1}} - \sqrt{a_{n-1}})(\sqrt{b_{n-1}} + \sqrt{a_{n-1}})$$

$$\leqslant ½|b_{n-1} - a_{n-1}|$$

Repeating this calculation we obtain eventually $b_n - a_n \leqslant |b-a|/2^n$.
The equal sign is only applicable if a = b when every a_k and b_k are also equal to this value.

318. Show how to place squares with sides of length (1/*m*) cm, where *m* = 2, 3, 4, 5, ... (an infinite number of them) inside a square with side of length 1 cm. None of the squares you use is allowed to overlap any other one.

Answer: The figure shows how a square of side 1 cm may be cut up into 2 squares of side ½ cm (making up ABGE), 4 squares of side ¼ cm (making up EFHD), 8 squares of side 1/8 cm (making up FGKJ), 16 squares of side 1/16 cm (making up JLMH), and so on. That this process can be continued indefinitely is clear enough, since each new batch of squares occupies a total of half the area occupied by the preceding batch, and uses up half of the remaining space. [For example, the next batch of 32 squares of side 1/32 cm (compared with the preceding batch, twice as many squares each of ¼ the area) would exactly fill out the lower half of the area LKCM.]



Now we place

the given squares of sides ½ and 1/3 in the two squares of side ½,
the given squares of sides ¼, 1/5, 1/6, 1/7 in the 4 squares of side ¼
the given squares of sides 1/8, 1/9, . . . 1/15 in the 8 squares of side 1/8.

and so on.

In general, the 2^n squares of sides $1/m$ with $2^n \leqslant m \leqslant 2^{n+1}-1$ are placed in the batch of 2^n squares of side $1/2^n$ in the figure.

Note: This problem shows that the infinite series $\Sigma 1/n^2 = 1 + ¼ + 1/9 + \ldots$ is less than 2. You might like to investigate its actual value.

319. A rectangular box has sides of length x cm, y cm and z cm where x,y,z are different numbers. The perimeter of the box is $p = 4(x+y+z)$, its surface area is $s = 2(xy+yz+zx)$ and the length of its main diagonal is $d = \sqrt{(x^2 + y^2 + z^2)}$. Show that the length of the shortest side is less than $[¼p-\sqrt{(d^2-½s)}]/3$ and the length of the longest side is greater than $[¼p+\sqrt{(d^2-½s)}]/3$.

Answer: First note that $(¼p)^2 = (x+y+z)^2 = d^2+s$ and so $d^2-½s = (¼p)^2-3s/2$. Let $3\alpha = ¼p-\sqrt{(d^2-½s)}$, $3\beta = ¼p+\sqrt{(d^2-½s)}$ and

$$f(X) = (3X-3\alpha)(3X-3\beta) = (3X-¼p)^2 - (d^2-½s) = ½(18X^2-3pX+3s)$$

This is a quadratic with roots α and β, and so $f(X) > 0$ if and only if $X < \alpha$ or $X > \beta$ (since $\alpha < \beta$).

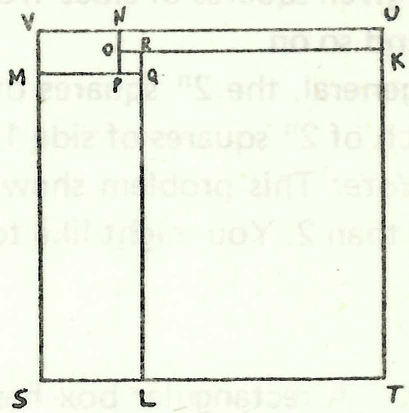
Now we may assume that $x < y < z$, and so $x < (x+y+z)/3 = p/12 = ½(\alpha+\beta) < \beta$.
Substituting for x, we get

$$\begin{aligned} f(x) &= 9x^2 - 6(x+y+z)x + 3(xy+yz+zx) \\ &= 3(x^2-xy-xz+yz) \\ &= 3(x-y)(y-z) > 0 \end{aligned}$$

and so $x < \alpha$ or $x > \beta$. But $x < \beta$ and so we must have $x < \alpha$. Similarly $z > \beta$.

320. A large square is divided into one small square (with side of length s cm) and four rectangles A, B, C and D which are not squares. No side of any rectangle is the same length as a side of another nor the side of the big square. The sides of A are 4s cm and 2s cm. B has the largest area of any of the rectangles. C has sides in the ratio 3:1 and its area is 300 sq. cm. Find the area of D.

Answer: One soon discovers that the only possible arrangement of the squares and rectangles must be as in the figure, with OPQR the small square of side s cm.



Let MPNV be the rectangle A. If *PM = 2s cm and *PN = 4s cm, we have *ON = 3s cm = *MQ which is inadmissible. Hence *PM = 4s cm and *ON = s cm.

Let *RK = x.s cm. Then *VU = (5+x).s cm and *MS = (3+x).s cm, *RL = (4+x).s cm. Which corner rectangle is C? It cannot be OKUN since then x = 2 cm and *RK = *PN. If LTKR is C we have (4+x):x = 3:1, again giving x = 2, which is inadmissible as before. Hence SLQM must be C, (3+x):5 = 3:1 whence x = 12. We see that the dimensions of C are 5s x 15s cm and since its area is 300 sq. cm s = 2. Note the dimensions of B(LTKR) are 12s x 16s or 24 x 32 cm and of D(OKUN) are s x 13s or 2 x 26 cm.

Hence D has an area of 52 sq. cm.

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Successful Solvers:

We apologise to those who had sent in solutions to previous problems for not having printed their names in the last issue. Below is a list of all solvers of problems this year:

P. Bos (Sydney Boys' High) 295, 306.
G. Clark (St. Joseph's) 289.
G. Elsworthy (Birrong Boys' High) 285, 286, 287, 288, 289.
A. Fekete (Sydney Grammar) 289, 290, 291, 292, 294, 295, 296, 305, 306, 307, 308, 317, 318, 319, 320.
A. Fisher (St. Ignatius) 297, 320.
L. Gale (Castle Hill High) 309, 310.
P. Haber (Sydney Grammar) 305, 307, 308.
R. Kidd (St. Joseph's) 309, 310.
H. Najna (Marist Brothers, Parramatta) 314, 316, 317, 319.
M. Nichols (C.E.G.G.S., Canberra) 285, 289.
J. O'Brien (St. Joseph's) 309, 310, 312.
M. Reynolds (Marist Brothers, Pagewood) 306, 307, 319, 320.
T. Rugless (St. Catherine's, Waverley) 297.
L. Rylands (Turramurra High) 289, 291, 293.
D. Swift (Sydney Technical High) 294, 295.