HOW TO CATCH REAL NUMBERS

What should we understand by a real number? Kronecker, a German Mathematician who flourished at the end of the last century, is supposed to have said: "God made the natural numbers: all else is the work of man". The integers and the rational numbers are comfortingly concrete and they would seem to be more than adequate for everyday use. This cosy view of the world was dented by Pythagorus and his school (5th century B.C.) when they discovered that $\sqrt{2}$ is irrational (Exercise: Why?) Yet $\sqrt{2}$ should be just as much of a number as 1. If I take the discrete, or engineering point of view, then $\sqrt{2}$ is what my electronic calculator says it is, viz 1.414213562. If I take the continuous point of view then $\sqrt{2}$ fits in some mysterious way into the line of rational numbers. Our satisfaction with this solution is shattered by Zeno (c. 450 B.C.). On the discrete view, motion is impossible because if you are at a particular point you cannot be moving. On the continuous view, motion is still impossible because you must always arrive at the half-way point before you reach your goal. The Greeks found more and more irrational numbers such as $\sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots \sqrt{17}$ (Theodorus, c. 450 B.C.), but they did not quite succeed in resolving what they were. They thought of numbers as lengths, products of 2 numbers as areas and products of 3 numbers as volumes. This made it almost impossible to do arithmetic, for example how many carpet tiles measuring 0.1 x 0.1 units are needed for a room measuring $\sqrt{2} \times \sqrt{3}$? Consequently, Greek science was largely qualitative. Their attitudes persist in our language (squares, cubes). Apparently, one poor fellow was sentenced to death under the Spanish inquisition for attempting to solve the quartic equation because it was a sin against God to contemplate x^4.

Stevin (1585) and Napier at about the same time hit on the idea of representing numbers by decimals. Given a real number, we set $a_0 = [x]$, the integer part of x, $x = a_0 + x_1/10$; $a_1 = [x_1]$, $x = a_0 + a_1/10 + x_2/10^2$; etc. That is $x = a_0.a_1\,a_2\ldots$ where the digits $a_1, a_2, \ldots$ are 0, 1, 2, . . . or 9. I wish to regard a real number as anything which can be written as $a_0 + a_1/10 + a_2/10^2 + \ldots = a_0.a_1\,a_2\ldots$ where the digits $a_1, a_2, \ldots$ are as before. We can make various observations about x from the shape of its decimal expansion.

First, the decimal may terminate, e.g. $0.1375 = 1375/10000 = 11/80 = 11/2^4 \times 5$. The numbers with finite decimal expansion are just the fractions $a/2^m 5^n$. Next, the decimal may become periodic, e.g.

$$\begin{aligned} 0.3\dot{4}\dot{5} &= 0.3454545\ldots \\ &= 3/10 + 45/1000 + 45/100000 + \ldots \\ &= \frac{3}{10} + \frac{45}{1000} \times \frac{1}{1-1/100} \\ &= 3/10 + 45/990 = 19/55. \end{aligned}$$

In fact, the numbers whose decimal expansions are finite or eventually periodic are just the rationals. For consider $x = m/n = a_0 + x_1/10$; $x_1 = 10(m-a_0 n)/n =$ $m_1/n = a_1 + x_2/10$; $x_2 = 10(m_1 - a_1 n)/n = m_2/n = a_2 + x_3/10; \ldots$ Since $m_1, m_2,$ $\ldots$ are integers (in fact multiples of 10) with $0 \leqslant m_j < 10n$, we must find that m_s $= m_{s+t}$ for some indices $s,t \geqslant 1$ and then $x = a_0.a_1.a_2 \ldots a_{s-1}\dot{a}_s a_{s+1} \ldots \dot{a}_{s+t-1}$, i.e. the decimal expansion of x is periodic. We have further that the maximum period length for m/n is n−1; this is attained for 1/7, 1/97 and others. By the way, $1/7 = 0.\dot{1}4285\dot{7}$, $2/7 = 0.\dot{2}8571\dot{4}$, $3/7 = 0.\dot{4}2857\dot{1}$, $4/7 = 0.\dot{5}7142\dot{8}$, $5/7 =$ $0.\dot{7}1428\dot{5}$, $6/7 = 0.\dot{8}5714\dot{2}$. Why?

A number whose decimal does not terminate or repeat must be irrational. For example, the following are irrational:

$$0.1010010001\ldots,\quad 0.1234567891011\ldots,\quad 0.23571113171923\ldots$$

Decimals have the advantage of ease of manipulation, i.e. there is a simple procedure for addition and multiplication. Their great disadvantage is that the decimal expansion of a real number depends on the accident that we have chosen the base 10. As a means of approximating a real number, the decimal expansion is unsatisfactory because it yields only fractions with denominators $2^m 5^n$. (why?)

As an historical example of this approximation problem, Huyghens (1660) set about constructing a model of the solar system by using toothed wheels. He had to determine what numbers of teeth for the wheels would give a ratio for 2 interconnected wheels as close as possible to the ratio α of the periods of revolution of the corresponding planets. At the same time, the number of teeth could not, for technical reasons, be too high. So we want to find a rational approximation p/q to α, with p,q not too large, which approximates α as closely as possible. We call p/q a best approximation to α if $|s\alpha - r| > |q\alpha - p|$ for every rational $r/s \neq p/q$ with $0 < s \leqslant q$.

If α and Q are any real numbers with $Q > 0$, we can find integers p and q such that $|q\alpha - p| \leqslant 1/Q$ and $1 \leqslant q < Q$. For we can suppose Q is an integer. Consider the distribution of the Q+1 points 1 and $\{j\alpha - [j\alpha] : 0 \leqslant j < Q\}$ among the Q intervals $\{x : k/Q \leqslant x \leqslant (k+1)/Q\}$ where $0 \leqslant k < Q$. At least one of the intervals must contain two of the points, so we can find integers r_1, r_2 s_1, s_2 such that $|(r_1\alpha - s_1) - (r_2\alpha - s_2)| \leqslant 1/Q$.

As a first best approximation to α we choose $p_0/q_0 = p_0/1$ where p_0 is the closest integer to α. If $\alpha = p_0/q_0$, the process stops. Otherwise, we can find p/q so that $|q\alpha - p| < |q_0\alpha - p_0|$. Let q_1 be the least q with this property and p_1 the corresponding p. If $\alpha = p_1/q_1$, the process stops. If not, we can continue the process to determine all the best approximations $p_0/q_0, p_1/q_1, p_2/q_2, \ldots$ to α in order of ascending q_n. If $\alpha = p/q$ is rational, the process clearly stops at some point $p_N/q_N = p/q$ since p/q is a best approximation. If α is irrational, the process continues indefinitely. Moreover, $|q\alpha - p| \geqslant |q_n\alpha - p_n|$ for $0 < q < q_{n+1}$, so $q_{n+1}|q_n\alpha - p_n| \leqslant 1$, i.e. $|\alpha - p_n/q_n| \leqslant 1/q_nq_{n+1}$, so $p_n/q_n \to \alpha$ as $n \to \infty$.

Let us find the best approximations to 157/68. (This procedure was used by the Hindus about 470 A.D.)

$157 = 2 \times 68 + 21$, $21 = 157 - 2 \times 68$:

$$p_0 = 2,\ q_0 = 1,\ p_0/q_0 = 2/1,\ q_0\alpha - p_0 = 21/68$$

$68 = 3 \times 21 + 5$, $5 = 7 \times 68 - 3 \times 157$:

$$p_1 = 7,\ q_1 = 3,\ p_1/q_1 = 7/3,\ q_1\alpha - p_1 = -5/68$$

$21 = 4 \times 5 + 1$, $1 = 13 \times 157 - 30 \times 68$:

$$p_2 = 30,\ q_2 = 13,\ p_2/q_2 = 30/13,\ q_2\alpha - p_2 = 1/68$$

$5 = 5 \times 1 + 0$, $0 = 157 \times 68 - 68 \times 157$:

$$p_3 = 157,\ q_3 = 68,\ p_3/q_3 = 157/68,\ q_3\alpha - p_3 = 0.$$

This is just the continued fraction algorithm:

$$\frac{157}{68} = 2 + \frac{21}{68} = 2 + \cfrac{1}{3 + \cfrac{5}{21}} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}$$

and the best approximations are given by deleting the remainders at each step. To save space (and typists) we will write this as $2 + \frac{1}{3+}\frac{1}{4+}\frac{1}{5}$. In general, any rational

p/q can be expanded as a continued fraction $\frac{p}{q} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\ldots\frac{1}{a_N}$ and the best approximations $\frac{p_n}{q_n} = a_0 + \frac{1}{a_1+}\ldots\frac{1}{a_n}$ are given by $p_0 = a_0$, $q_0 = 1$; $p_1 = a_0a_1 + 1$, $q_1 = a_1$; $p_{n+1} = a_{n+1}p_n + p_{n-1}$, $q_{n+1} = a_{n+1}q_n + q_{n-1}$.

The continued fraction provides the best approximations for any real number a. We proceed as follows:

$$a = a_0 + 1/x_1,\ a_0 = [a];\ x_1 = a_1 + 1/x_2,\ a_1 = [x_1];\ x_2 = a_2 + 1/x_3,\ a_2 = [x_2];\ \text{etc.}$$

Thus $a = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\frac{1}{x_n} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\ldots$ For example,

$$\pi = 3.141592654\ldots = 3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{14+}\frac{1}{2+}\frac{1}{1+}\ldots$$

$n = 0$	1	2	3	4	5	6	7	8	9
$a_n = 3$	7	15	1	292	1	1	1	2	1
$p_n = 3$	22	333	355	103993	104348	208341	312689	833719	1146408
$q_n = 1$	7	106	113	32996	33109	66105	99214	264533	363747

Observe that $|\pi - 355/113| = |\pi - p_3/q_3| < 1/q_3q_4 = 1/113 \times 32996 < 3\times10^{-7}$. Apparently, there is no pattern to the a_n, but Brouncker (1655) observed that $\frac{4}{\pi} = 1 + \frac{1}{2+}\frac{9}{2+}\frac{25}{2+}\frac{49}{2+}\ldots$ Consider another example:

$$a = \sqrt{2} = 1 + 1/x_1;\ x_1 = 1/(\sqrt{2}-1) = \sqrt{2} + 1 = 2 + 1/x_2;\ x_2 = 1/(\sqrt{2}-1) = x_1;\ldots$$

Thus $\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\ldots$ It can be proved that any irrational number of the form $(a + b\sqrt{d})/c$ with a, b, c, d integers has a periodic continued fraction. The converse is also true. For example, consider $a = 3 + \frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\ldots$

Then $a - 3 = \frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\ldots = \frac{1}{2+}\frac{1}{1+a-3} = \frac{a-2}{2a-3}$, i.e. $2a^2 - 10a + 11 = 0$, so $a = (5 + \sqrt{3})/2$.

Suppose now that we wish to find a positive integer n such that n+1 and ½n+1 are both perfect squares. We require $n+1 = x^2$, $\frac{1}{2}n+1 = y^2$, so $x^2-2y^2 = -1$. This says $|x-y\sqrt{2}| = 1/(x+y\sqrt{2})$ so we might expect x/y to be a best approximation to $\sqrt{2}$.

Using the continued fraction expansion for $\sqrt{2}$, we get

n	=	0	1	2	3	4	5	6
a_n	=	1	2	2	2	2	2	2
p_n	=	1	3	7	17	41	99	239
q_n	=	1	2	5	12	29	70	169
$p_n^2-2q_n^2$	=	−1	1	−1	1	−1	1	−1
p_n^2-1	=	0	.	48	.	1680	.	57121

and infinitely many more.

The equation $x^2-dy^2 = \pm 1$ was (wrongly) called Pell's equation by Euler after Brouncker and Wallis had given a partial solution (Euler thought Pell had done it). It can always be solved using best approximation in this way. It was known much earlier to the Hindus. Brahmagupta (650 A.D) wrote: "A person who can within a year solve the equation $x^2-92y^2 = 1$ is a mathematician". The smallest solution is $x = 1151$, $y = 120$.

Even earlier, Archimedes (400 B.C.) perpetrated his famous cattle problem:

> Compute O friend, the host of the oxen of the sun, giving thy mind thereto, if thou hast a share of wisdom. Compute the number which once grazed upon the plains of the Sicilian Isle Trinacia and which were divided according to color into four herds, one milk white, one black, one yellow and one dappled. The number of bulls formed the majority of the animals in each herd and the relations between them were as follows: [We write W, B, Y, D for the respective numbers of bulls and w, b, y, d for the cows] $W = 5B/6 + D$, $B = 9Y/20 + D$, $Y = 13W/42 + D$, $w = 7(B+b)/12$, $b = 9(Y+y)/20$, $y = 11(D+d)/30$, $d = 13(W+w)/42$.
>
> If thou canst give, O friend, the number of bulls and cows in each herd, thou art not unknowing nor unskilled in numbers, but still not yet to be counted among the wise. Consider however, the following additional relations between the bulls of the sun. W+B is a square, and Y+D is a triangular number. When thou hast then computed the totals of the herds, O friend, go forth as conqueror, and rest assured that thou art proved most skilled in the science of numbers.

This is equivalent to solving $x^2-4729494y^2 = 1$; we refer the reader to Chapter 22 of Albert H. Beiler's excellent paperback "Recreations in the theory of numbers. The Queen of Mathematics entertains" printed by Dover books. That this is an interesting problem may be gathered from the fact that it was completely solved by computer only a few years ago, with the individual numbers being about a third of a mile long!

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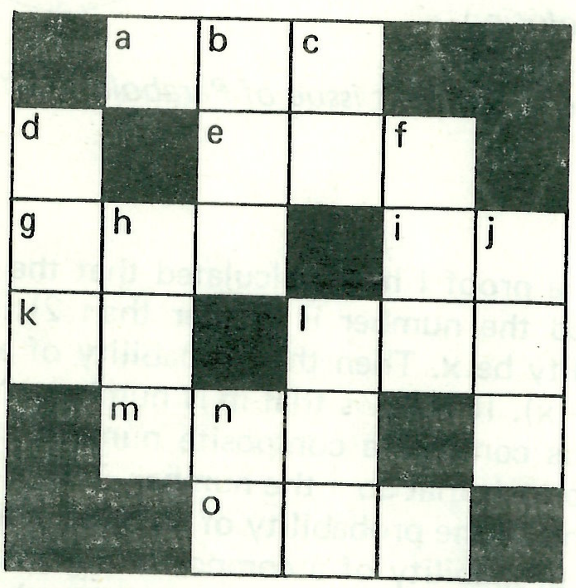
Crossnumber
(adapted from Mathematical Digest, New Zealand)

A, B, C, D, E are positive whole numbers whose values may be deduced.
The first digit of each number in the crossnumber is not 0.



Across
(a) $(2A + B)(A-B)$
(e) $10(A + B) + 1$
(g) E^2
(i) BC
(k) $A + B + C + D$
(l) Same as (f) down
(m) $(C + D)E$
(o) $11(A + B + C + D + E)$

Down
(b) A multiple of C
(c) C^2
(d) $100 + E$
(f) Same as (l) across
(h) $10(B + D)^2 + 1$
(j) 274D
(l) 5CD
(n) 2E

If you find it difficult, see the hint on page 22.