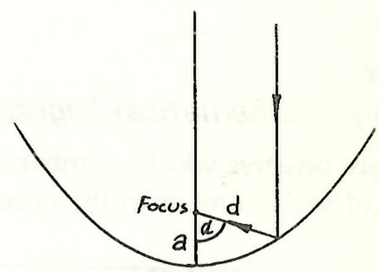
YOUR LETTERS

Dear Sir,

The following "real life" problem was given to me by a student at North Sydney Tech. College (where I teach) only a few weeks ago, and it might be suitable for Parabola. This fellow wanted a Parabolic reflector constructed. The length d and the angle α were specified. He wondered if there was an easy way to find the focal length a.

Mr G. Davis



[Rather than print Mr Davis' solution, I thought I would ask you to send in solutions and print the best one in the next issue of Parabola – Ed.]

Dear Sir,

I would like to show you a proof I have calculated that the probability of a number being prime (provided the number is greater than 2) is 1/3. My proof starts by letting the probability be x. Then the probability of a number greater than 2 being composite is $(1-x)$. It follows that in N numbers, Nx are prime and $N(1-x)$ are composite. Let us consider a composite number greater than 2. It must have two factors a and b such that ab = the number. The probability of each of a and b being odd is ½, and so the probability of ab being odd (i.e. both a and b being odd) is ¼. Thus the probability of a composite number being odd is ¼, and the probability of it being even is ¾. Now, we know that the probability of a number being odd = the probability of a number being even = ½. We also know:

(1) all even numbers greater than 2 are composite, and ¾ of all composites are even,

(2) all odds are either prime or composite, and ¼ of all composites are odd.

Using the facts and figures shown above, it is obvious that

$$\text{Number of evens} = \tfrac{3}{4}N(1-x) = \tfrac{1}{4}N(1-x) + Nx = \text{number of odds}$$

The N's cancel, and we get a simple equation

$$3(1-x) = 1-x+4x = 3x+1$$

with the only value of x being 1/3

Now, I would like to disprove this by another method. It is obvious that ½ of all numbers are divisible by 2. Of the remaining numbers 1/3 are divisible by 3, and so 1/6 of all numbers are divisible by 3, but not by 2. Adding: 2/3 of all numbers are divisible by 2 or 3.

Of all the numbers greater than 2 which are divisible by 2 or 3, only one (3) is prime. Thus we only have to prove that there is more than one composite number not divisible by 2 or 3 to show that there cannot be as many as 1/3 of all numbers being prime (i.e. that $x < 1/3$). Two such numbers are 25 and 35.

This proves that my earlier proof is wrong!

I would appreciate it if you could throw some light on this matter.

Philip Stott

Editor's Reply:

Your question can be answered using the sieve of Eratosthenes. This is a device for making a list of primes. Say, we want a list of the primes less than 100. Make a list of all the numbers up to 100:

2, 3, 4, 5, 6, 7, 8, 9, 10, . . . , 99, 100.

(Ignore 1, which we do not consider to be a prime.) Take the first number in the list, namely 2, as the first prime. Now cross out 2 and all its multiples in the list:

~~2~~, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, . . . , 99, ~~100~~.

Take the first number which remains, namely 3, as the next prime and cross out 3 and all its multiples in the list:

~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, . . . , ~~99~~, ~~100~~.

The first number which remains, namely 5, is the next prime and so on. Now any composite number in the original list must have at least one prime factor less than 10, so after we have crossed out all the multiples of all the primes less than 10, anything that remains must be prime. Try it and see.

Now let us try to interpret this probabilistically. At the first step, we cross out every second number, which leaves 100(1−½) numbers. At the second step, we cross out every third number from what remains, leaving 100(1−½)(1−1/3) numbers approximately. This is so because the events "2 divides a number" and "3 divides a number" are approximately independent. Continuing in this way, after dealing with the multiples of all primes less than 10, we will have approximately

$$100(1-\tfrac{1}{2})(1-1/3)(1-1/5)(1-1/7) \cong 23$$

numbers left. To get an estimate for the number of primes less than 100, we have to add on the number of primes we have discarded, that is 2, 3, 5 and 7 and subtract off 1 since we do not count the number 1 as a prime. This gives us that the number of primes less than 100 is approximately 23 + 4−1 = 26. We are only

1 out! Note that the probability that one of the numbers less than 100 is prime is approximately

$$(1-½)(1-1/3)(1-1/5)(1-1/7) \cong 0.23$$

We can adapt this process to make a list of all primes less than N, say. Now we must apply the crossing out process to the multiples of all the primes less than $\sqrt{N}$; say these are 2, 3, 5, 7, . . . , p. Then the probability that one of the numbers less than N is prime is approximately

$$(1-½)(1-1/3)(1-1/5)(1-1/7)\ldots(1-1/p).$$

It can be shown that this works out to be about $\frac{1}{\log N}$ for large N. To put it another way, the number of primes less than N is about $\frac{N}{\log N}$. Observe that the probability of finding a prime among the numbers 1, 2, . . . , N by picking a number at random gets smaller and smaller as N gets larger. You see that your second proof is on the right track; it corresponds to taking account of the multiples of 2 and 3 in the sieve process.

Your first proof shows just how careful you need to be in discussing probabilities to specify exactly what you are counting. As you observe, if a and b are picked at random, then the product ab has probability ¾ of being even and probability ¼ of being odd. Now every number n can be written as a product n = ab, but in counting probabilities in this way, we count all these factorisations ab and not just all the numbers n. For example,

$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4 = 4 \times 3 = 6 \times 2 = 12 \times 1$$

is counted 6 times. Your argument shows that, on the average, an even number has more factorisations than an odd number, as you might perhaps expect.

Dear Sir,

Unable to study, I let my mind wander onto the Fibonacci system. I ran quickly through it a while before writing it out modulo nine (9). This is the result:

1,1,2,3,5,8,4,3,7,1,8,0,8,8,7,6,4,1,5,6,2,8,1,0, (1,1,2,3,5, . . .)

As you notice, it begins all over again. Whilst toying with it, I noticed a curious pattern. The 2nd digit equals the 2nd last one. The fourth equals the 2nd last + 4th last. The 6th = 2nd last + 4th last + 6th last and the 8th = (2nd last + 4th last + 6th last + 8th last) modulo 9. Having no time left (I wrote this letter during French exam.) I didn't continue. Perhaps you could enlighten me as to the reason this works and any other patterns contrived from Fibonacci modulo 9 (or any other moduli).

J. O'Brien

Editor's Reply:

Far be it from me to encourage anyone to investigate Mathematical curiosities during French exams, but I will endeavour to answer your questions:

(a) If m is any positive integer, the Fibonacci sequence must repeat modulo m after r steps where $r \leqslant m^2$. This is because as soon as any pair of consecutive terms recurs, the whole sequence must recur. Thus, if we write F_n for the n'th term and

$$F_{r+n} \equiv F_n,\ F_{r+n+1} \equiv F_{n+1} \pmod m \text{ then}$$
$$F_{r+n+2} = F_{r+n} + F_{r+n+1} \equiv F_n + F_{n+1} \pmod m = F_{n+2} \text{ etc.}$$

If r is the smallest positive integer for which this works, we say the series has period r modulo m.

The value of r cannot be greater than the number of possible values, modulo n, of the ordered pair (F_n, F_{n+1}), and this number is m^2 since there are m possible values for each of F_n and F_{n+1}. (This argument is also used in the solution to problem 331).

(b) The pattern you noticed is also true modulo m for any value of m. Let the length of the period modulo m be r (and so, in your case, m = 9 and r = 24). What we are required to show is that, modulo m,

(1) $F_{2n} \equiv F_{r-1} + F_{r-3} + \ldots + F_{r-2n+1}$

Now, this is certainly true for n = 1, since $F_2 = 1 \equiv 1$ mod m. (We know $F_{r+1} \equiv 1, F_{r+2} \equiv 1$, and, working backwards, $F_r \equiv 0$, $F_{r-1} \equiv 1$.)

Suppose we knew that

(2) $F_{2n+1} \equiv F_{r-(2n+1)}$

Adding (1) and (2) we would have

$$F_{2n} + F_{2n+1} \equiv F_{r-1} + F_{r-3} + \ldots + F_{r-2n+1} + F_{r-2n-1},$$
or
$$F_{2n+2} \equiv F_{r-1} + F_{r-3} + \ldots + F_{r-(2n+2)+1}.$$

All that remains is to prove (2). Indeed we show that

(3) $F_n \equiv (-1)^{n+1} F_{r-n}$,

of which (2) is the case of n odd.

(3) is true for n = 1, n = 2. ($F_1 = 1$, $F_{r-1} \equiv 1$; $F_2 = 1$, $F_{r-2} = F_r - F_{r-1} \equiv 0 - 1 = -1 = -F_2$)

and the inductive step is as follows:

$$
\begin{aligned}
F_{n+1} &= F_{n-1} + F_n \\
&= (-1)^n F_{r-(n-1)} + (-1)^{n+1} F_{r-n} \\
&= (-1)^n F_{r-n+1} + (-1)^{n+1} F_{r-n} \\
&= (-1)^n [F_{r-n+1} - F_{r-n}] \\
&= (-1)^n F_{r-n-1} \\
&= (-1)^{(n+1)+1} F_{r-(n+1)} \quad \text{(Note that } (-1)^2 = 1\text{)},
\end{aligned}
$$

completing the proof.

Maybe you, or some other readers, might like to supply us with more patterns in the Fibonacci sequence. This sequence is

$$\{1, 1, 2, 3, 5, 8, \ldots\}$$

in which each term F_{n+2} is the sum of the two preceding terms $F_n + F_{n+1}$.

Dear Sir,

As a result of N.G. Serafim's letter in your last issue and question 2 in the Senior Division of the Mathematics Competition of 1976, I have looked into the possibility of more than six primes in Arithmetic Sequence. Using simple Diophantine Equations and a small amount of Number Theory, I have come up with ten primes in Arithmetic Sequence with a difference of 210 (= 2 x 3 x 5 x 7). They are:

199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089.

Note that this is the greatest number of primes in A.P. with a difference of 210, because 11 divides every eleventh member of such a series. This can be seen by writing a for the first term of the series and choosing the smallest number n so that 11 divides n + a (in the given sequence, a = 199 and n = 10). Since 11 divides 209n, it also divides the term 210n + a = 209n + (n + a). Similarly, 11 divides 210(n + 1) + a, 210(n + 2) + a, etc and so divides every eleventh term.

Do you know what is the greatest number of primes in Arithmetic Sequence discovered so far?

Lindsay Kleeman.

Editor's Reply:

The longest Arithmetic sequence of prime numbers I know is:

4943, 65003, 125063, 185123, 245183, 305243, 365303, 425363, 485423, 545483, 605543, 665603, 725663.

This sequence was found a few years ago by a man called Seredinsky. Before that your sequence was the longest known. (It was found by V. Thébault in 1944.)

To find a longer sequence, let us denote the sequence by

$$p, p+d, p+2d, \ldots p+(n-1)d,$$

where there are n terms in the sequence (and so, in your example $n = 10$, $p = 199$, $d = 210$ and in Seredinsky's sequence $n = 13$, $p = 4943$, $d = 60060$). If q is a prime number not dividing d, then (by the same argument as you have given) it is easy to show that q divides every q'th member of the series and so either $n < q$ or q is one of the terms of the series. In the second case, $n \leqslant q$.

For example, if d is odd then 2 is a prime number which does not divide d, and so $n \leqslant 2$ and the Arithmetic sequence must only have 1 term except for the sequences 2, 3 or 2, 5, . . . or 2, p where p is any prime. Thus, to get a sequence with more than 2 terms, d must be even. Similarly, to get a sequence with more than 3 terms, d must be divisible by 3 and also by 2 (the sequence must have more than two terms as well!): thus d must be divisible by 6.

In general, the common difference d will have to be divisible by the first k primes $p_1, p_2, \ldots p_k$ in order to yield a sequence with more than p_k terms. In Seredinsky's example, d is divisible by the prime numbers $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$ and $p_6 = 13$. By repeating this argument up to $p_{25} = 97$, we can see that if we wanted an Arithmetic sequence of 100 prime numbers, d would have to have at least thirty digits!

The obvious question to ask now is: Is it possible to go on finding longer and longer Arithmetic sequences of prime numbers? Some Mathematicians believe it is, and have suggested other questions:

(1) If d is divisible by $p_1 \, p_2 \ldots p_k$ but not divisible by p_{k+1}, is there an Arithmetic sequence

$$p, p+d, \ldots p+(n-1)d$$

where $n = p_{k+1}-1$? (We have shown $n \leqslant p_{k+1}-1$).

(2) There are an infinite number of examples, such as 3, 5, 7 or 7, 13, 19 or 3, 13, 23, of Arithmetic sequences of three prime numbers. Are there an infinite number of examples of Arithmetic sequences of n prime numbers for other values of n?

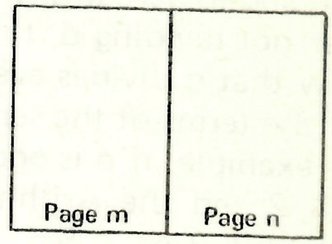
(3) Are there an infinite number of examples of sequences p, p+2 of primes (called twin primes)? If so, are there an infinite number of examples of sequences p, p+2, p+6, (Note: This is not an Arithmetic sequence!) In general, is there a set of numbers $a_1, a_2, \ldots, a_n$ so that there are an infinite number of examples of sequences $p, p+a_1, p+a_2, \ldots p+a_n$ of prime numbers. Some Mathematicians believe that the answer to all the above questions is yes. But be warned – the Mathematics that would be needed to prove them would be very difficult!

As you can see, your question is very stimulating and far-reaching.

If you are interested in reading further on these topics, an excellent book is "Elementary Theory of Numbers" by W. Sierpinski (a Polish Mathematical monograph) where I myself obtained most of the above information.

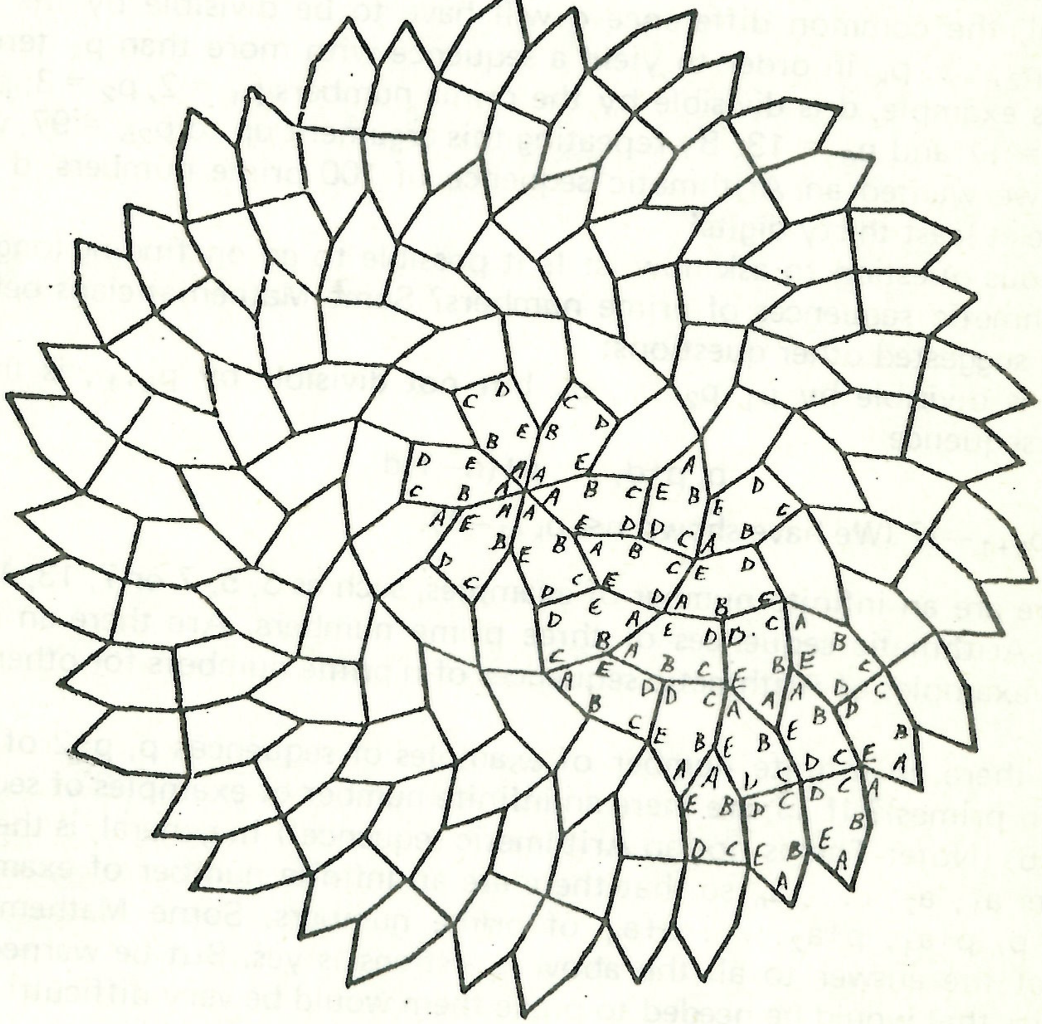
An Imposing Problem

When preparing Parabola for printing, the material for each page is assembled onto separate sheets of paper. The final step is to "impose" the pages, that is paste pairs of them together so that when printed the pages appear in the correct order.



The figure shows pages numbered m and n, ready for printing. Devise a simple rule that the printer can use to check that these pages are imposed correctly.

(An answer, barring printing mishaps, may be found on page 22).



$6A = 360^o$, $A + B + E = 360^o$, $2A + D + E = 360^o$, $A + 2C + E = 360^o$, $C + 2E = 360^o$, $B + 2D = 360^o$, $2C + 2D = 360^o$ $A = 60^o$, $B = 160^o$, $C = 80^o$, $D = 100^o$, $E = 140^o$ (See page 2)