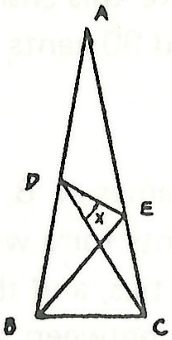
343. We define a "shuffle" of a deck of N cards numbered 1, 2, . . . , N to be a specific procedure for arranging them in a different order. If one systematically repeats the same shuffle of the deck enough times, it returns to its original order. For instance, if the shuffle consists of interchanging the top 2 cards of the deck it returns to its original order after 2 shuffles. What shuffle of a deck of 28 cards requires the largest number of repetitions before returning to the original order?

344. *(Suggested by Mr G. Davis of North Sydney Tech. College).* In the following diagram *AB = *AC, ∠DAE = 20°, ∠DCB = 60°, ∠EBC = 50° and ∠CDE = x°. Find x, without using trigonometric tables.



Solutions to Problems 321—332 (Vol. 12 No. 3)

321. The following factorisations of numbers are true:

$$12 = 3.4;\quad 1122 = 34.33;\quad 111222 = 334.333;\quad 1111222 = 3334.3333$$

Can this scheme be continued indefinitely? Prove your answer.

Answer: Suppose A = 33 . . . 33 and B = 33 . . . 34 (each one having n digits). Then $3A = 99\ldots 99 = 10^n - 1$, $3B = 100\ldots 02 = 10^n + 2$, and

$$\begin{aligned} 9AB = (3A)(3B) &= (10^n - 1)(10^n + 2) \\ &= 10^{2n} + 10^n - 2 \\ &= 10^{2n} - 1 + 10^n - 1 \\ &= 99\ldots 999\ldots 99 + 99\ldots 9 \end{aligned}$$

where the first number has 2n nines and the second has n nines. So AB = 11 . . . 111 . . . 1 + 11 . . . 1 = 11 . . . 122 . . . 2.

322. Suppose there were 250,000 people in Sydney in 1968 who made between \$8,000.00 and \$9,000.00. Show there were at least 3 people who made the same salary down to the last cent.

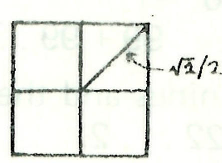
Answer: Let us suppose instead that no more than 2 people made the same salary. As there are only 100,000 different possible salaries between \$8,000.00 and \$9,000.00, this would mean that no more than 200,000 people made between those two salaries. This would leave a remainder of 50,000 people not accounted for, and so could not possibly be true. Hence at least one salary occurred more than twice. (To make this easier to see, change the problem to 25 people earning between 80 cents and 90 cents and try this.)

323. Twenty-six entrants, with names A, B, C, . . . , Z, play a chess tournament, each against all others. Score 2 points for a win, 1 for a draw, and 0 for a loss. No one's total was odd, there were no ties, and they ended in the order A, B, C, . . . , Z. What was the result of the match between M and N? Prove it.

Answer: A player has a maximum possible score of 50 points, if he wins every game. As there are only 26 even numbers 0, 2, 4, 6 . . . 48, 50, each of these scores must have resulted (since no two players tied with the same score). Hence A scored 50 points and won every game, B scored 48 points and won every game except against A, and so on. (Assuming the leading k players beat everyone below them, the (k+1)'th player, with only one less win than the k'th, must also have beaten everyone finishing below him.) Hence M beat N.

324. Suppose that five points are located in a square of side length 1. Prove that at least two of the points must be within $\sqrt{2}/2$ of one another.

Answer: *(Submitted by Glenn Reeves)* Divide the square into 4 quarters. Obviously no 2 points may be placed in a quarter so that they are greater than $\sqrt{2}/2$ apart, since the diagonal, the greatest possible distance between 2 points in a square, is equal to $\sqrt{2}/2$.



Since there are 4 quarters and 5 points, two points must be placed in one of the quarters, thus placing 2 points within $\sqrt{2}/2$ of one another.

325. The sides of a triangle are a, b, c units where a, b, c are integers and $a \leqslant b \leqslant c$. If c is given, show that the number of different triangles is $\frac{1}{4}(c+1)^2$ or $\frac{1}{4}c(c+2)$ according as c is odd or even.

Answer: In this problem, we use the fact that the sum of the lengths of two sides of a triangle is greater than the length of the third side, and so $2b \geqslant a+b > c$.

Case 1. If c is even, then $c = 2k$, where k is an integer. Since $2c \geqslant 2b > c$, we must have $2k \geqslant b > k$ and so $b = k+1, k+2 \ldots,$ or $2k$. If we write $b = k + r$ (where $r = 1, 2, \ldots,$ or k) then, since $c-b < a \leqslant b$, we have $k-r < a \leqslant k+r$ and so $a = k-r+1, k-r+2, \ldots,$ or $k+r$. For each value of r we have 2r possible values for a and so the number of possible values for both a and b is

$$2\times1 + 2\times2 + 2\times3 + \ldots + 2\times k = k(k+1) = \tfrac{1}{4}c(c+2).$$

Case 2. If c is odd, then $c = 2k+1$, where k is an integer. As before, we see that $b = k+1, k+2, \ldots$ or $2k+1$, and (writing $b = k+r$ again) the allowable values of a are $k+2-r, k+3-r \ldots, k+r$. For each value of r, we have $2r-1$ possible values for a and so the number of possible values for both a and b is

$$(2\times1-1) + (2\times2-1) + \ldots + (2(k+1)-1) = (k+1)(k+2)-(k+1) = (k+1)^2 = \tfrac{1}{4}(c+1)^2.$$

326. Suppose that mn boys are standing in a rectangular formation of m rows and n columns. Suppose that the boys in each row get shorter going from left to right. Suppose someone rearranges each column, independently of one another, so that going from front to back the boys get shorter. Show that the boys in each row still get shorter going left to right.

Answer: If, after rearranging, boys A and B are both in the ith row and B is to the right of A, then we wish to show that A's height, a, is greater than B's height, b. Suppose the contrary is true. Then B and all in front of him in the same column (i boys in all) are taller than A and all behind him in his column. Hence only the $i-1$ boys (at most) in front of A are taller than the i boys from B forward in B's column. But this contradicts the fact that each of the boys in B's column was formerly paired with a taller boy in A's column. Hence the supposition is false.

327. If a pack of playing cards is shuffled systematically and the operation of shuffling repeated exactly, then after a certain number of repetitions of the operation, the original order of the pack will be restored. Suppose the pack is shuffled as follows: Hold the pack face down in the left hand; in the right hand, take the top half of the pack and insert it into the lower half so that each right-hand card is above the corresponding left-hand card.

(a) After how many shuffles is a 52-card pack returned to order?

(b) After how many shuffles is a 26-card pack returned to order?

Answer: (a) Label the cards 1, 2, 3, . . . , 52 so that card n is n'th from the top of the original pack. Note that card 1 stays on the top of the pack after any number of shuffles, and likewise card 52 stays on the bottom. After one shuffle card 18 is in the position previously occupied by card 35 and card 35 is in the position previously occupied by 18. Hence after 2 shuffles (or any even number of shuffles) these cards will be in their original places. The remaining 48 cards fall into six 8-card cycles of which a typical one is 2, 3, 5, 9, 17, 33, 14, 27, (2). After one shuffle each of these cards occupies the position previously occupied by the next one in the list. (For example, card 33 will be in 14th place.) Hence after 2 shuffles a card will occupy the position originally occupied by the card next but one in the list (e.g. card 33 will be in 27th place). After 8 shuffles all of these cards will be in their original places. You may check, if you wish, that the other 8-card cycles are

(7, 13, 25, 49, 46, 40, 28, 4, (7)).
(6, 11, 21, 41, 30, 8, 15, 29, (6)).
(10, 19, 37, 22, 43, 34, 16, 31, (10)).
(12, 23, 45, 38, 24, 47, 42, 32, (12)) and
(20, 39, 26, 51, 50, 48, 44, 36, (2)).

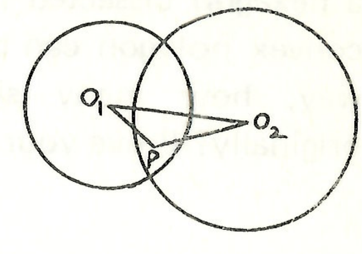
Hence after 8 shuffles, every card in the pack occupies its original position.

(b) Label the cards in the original deck 1, 2, 3, . . . 26 from the top down. Cards 1 and 26 never change their position. Cards 6, 11, 21, 16 (6) form a 4 card cycle, and the remaining 20 cards (2, 3, 5, 9, 17, 8, 15, 4, 7, 13, 25, 24, 22, 18, 10, 19, 12, 23, 20, 14, (2)) form a cycle. Hence after 20 shuffles every card is in its original position. In general, the number of shuffles required is the least common multiple of the lengths of all the cycles.

You might like to suggest the answer to this problem if there are n cards in the pack for values of n other than 52 or 26.

328. Six circular areas are lying in the plane so that no one of them covers the center of another. Show that there is no point in common to all six circular areas.

Answer: Suppose on the contrary that there is a point P which lies inside each of the six circles and let the centres of the circles be $O_1, O_2, O_3, O_4, O_5, O_6$. Since P is in the first 2 circles, $*O_1P < r_1$ and $*O_2P < r_2$ (where r_1, r_2 are the radii of the 2 circles). Since O_2 is not inside the first circle, $*O_2O_1 > r_1 > *O_1P$ and similarly $*O_2O_1 > *O_2P$. Thus O_2O_1 is the longest side of the triangle O_1O_2P, whence $\angle O_1PO_2$ exceeds $60°$ (you might like to check this fact!).



Similarly $\angle O_2PO_3, \angle O_3PO_4, \angle O_4PO_5, \angle O_5PO_6, \angle O_6PO_1 > 60°$ and so their sum is greater than $6 \times 60° = 360°$ which is impossible.

329. Consider the following array of natural numbers similar to Pascal's triangle. If we denote the nth row of the triangle by $a_{n1}, a_{n2}, a_{n3}, \ldots, a_{n,n-1}, a_{nn}$, then the law of formation is given by

$$a_{n1} = a_{nn} = 1$$

and for $2 \leqslant i \leqslant n-1$

$$a_{n,i} = (n-i+1)a_{n-1,i-1} + ia_{n-1,i}.$$

n									
1					1				
2				1		1			
3			1		4		1		
4		1		11		11		1	
5	1		26		66		26		1

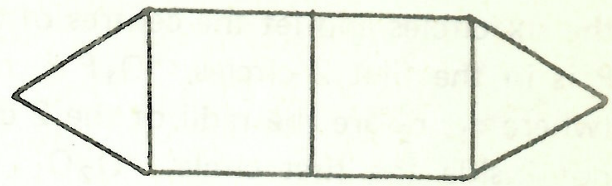
For example, $a_{5,2} = (5-2+1)a_{4,1} + a_{4,2} = 4.1 + 2.11 = 26$, and $a_{5,3} = (5-3+1)a_{4,2} + 3a_{4,3} = 3.11 + 3.11 = 66$. (You should construct rows 6, 7, and 8 to understand the method of construction.) Find a simple formula involving n, for the sum s_n of the nth row, $s_n = a_{n1} + a_{n2} + a_{n3} + \ldots + a_{nn}$ and prove it. ($s_1 = 1$, $s_2 = 2$, $s_3 = 6, \ldots$).

Answer: The sum $S_n = n!$ Observe that

$$\begin{aligned} S_n &= 1 + a_{n,2} + a_{n,3} + \ldots + a_{n,n-1} + 1 \\ &= 1 + [(n-1)a_{n-1,1} + 2a_{n-1,2}] + [(n-2)a_{n-1,2} + 3a_{n-1,3}] + \ldots \\ &\qquad + [2a_{n-1,n-2} + (n-1)a_{n-1,n-1}] + 1 \\ &= 1 + (n-1)a_{n-1,1} + (2+n-2)a_{n-1,2} + (3+n-3)a_{n-1,3} + \ldots + (n-2+2)a_{n-1,n-2} \\ &\qquad + (n-1)a_{n-1,n-1} + 1 \\ &= n(1 + a_{n-1,2} + a_{n-1,3} + \ldots + a_{n-1,n-2} + 1) \\ &= n\,S_{n-1}. \end{aligned}$$

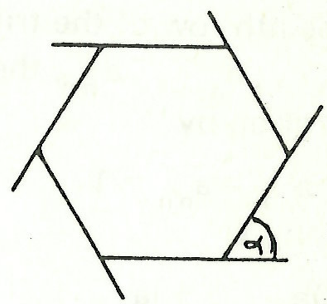
Since $S_1 = 1!$, the value of S_n follows by mathematical induction.

330. Certain convex polygons can be dissected into squares and equilateral triangles all having the same length of side. For example, the illustration shows a hexagon dissected in such a way. If a convex polygon can be dissected in this way, how many sides did it have originally? Prove your answer.

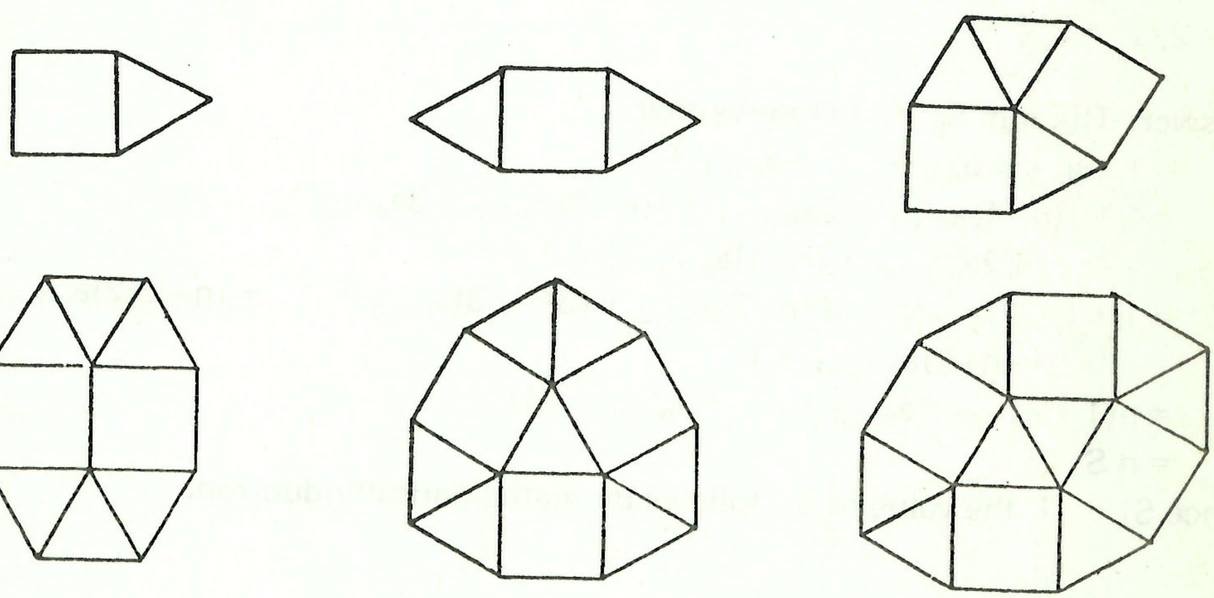


Answer: The interior angles of the polygon are made up from angles of size 60° or 90°. Thus their sizes are 60°, 90°, 120° or 150° and so the exterior angles (such as α in figure 1) are 30°, 60°, 90° or 120°. Since the sum of the exterior

Figure 1



angles is 360° it is clear that the largest number of sides possible is 12. A little experimentation enables one to construct polygons with 5, 6, 7, 8, 9, 10, 11 or 12 sides. For example



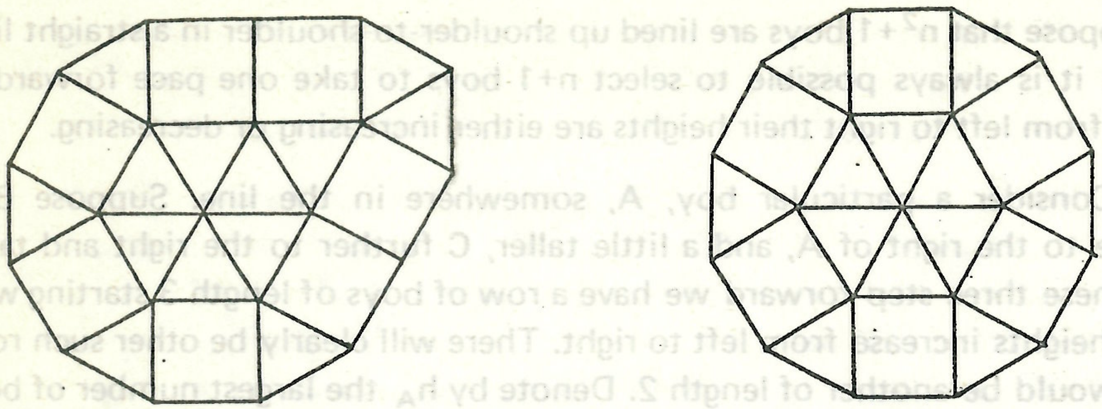


Figure 2

Neither any triangle, nor any quadrilateral, can be dissected into a mixture of squares and equilateral triangles. The triangle must have all its angles 60°. The corner piece at vertex A must be a triangle (see figure 3), but then the angles ∠CBX and ∠BCY are both 120°, which forces another layer of triangles DEB, BEC, CEF. This argument may be repeated for the angles FDX and DFY, and so on, until the whole figure is covered with triangles.

For a quadrilateral, if the angles are all right angles a similar argument shows that its dissection contains only squares. If the angles are two 60° and two 120°, again a similar argument shows that only triangles can occur. However, if the angles are 90°, 90°, 120°, 60° or 90°, 150°, 60°, 60° no dissection is possible. For let G be an angle of 90° in the quadrilateral (see figure 4). The corner piece at G must be a square. A row of identical squares must take us to each of the neighbouring vertices H and K of the quadrilateral. We observe that ∠H and ∠K must both be either 90° or 150°.

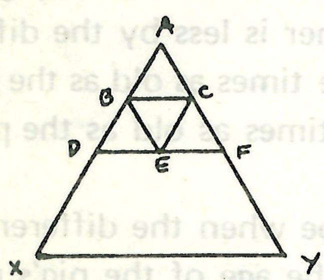


Figure 3

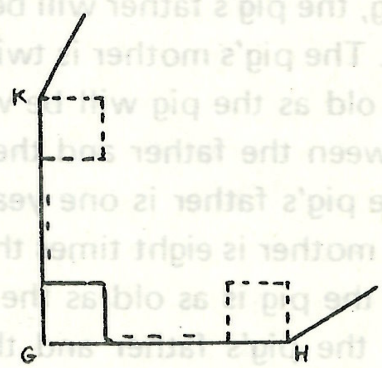


Figure 4

331. Suppose that n^2+1 boys are lined up shoulder-to-shoulder in a straight line. Show that it is always possible to select $n+1$ boys to take one pace forward so that going from left to right their heights are either increasing or decreasing.

Answer: Consider a particular boy, A, somewhere in the line. Suppose B is somewhere to the right of A, and a little taller, C further to the right and taller again. If these three step forward we have a row of boys of length 3 starting with A, whose heights increase from left to right. There will clearly be other such rows e.g. A, C would be another of length 2. Denote by h_A the largest number of boys in such a row starting at A whose heights increase from left to right. Similarly, let k_A denote the largest number of boys in a row starting with A and with heights decreasing steadily from left to right. In this way we have associated with A an ordered pair of natural numbers (h_A, k_A). For example, the boy at the far right of the row is "represented" by the ordered pair (1,1), and the boy second from the end by either (2,1) or (1,2) according as he is shorter or taller than the end boy.

Observe that if X and Y are any two different boys in the original line, then (h_X, k_X) and (h_Y, k_Y) cannot be identical pairs. For definiteness assume Y is to the left of X. Then if Y is shorter than X, h_Y must be greater than h_X. [For suppose we can find a row of h_X boys commencing with X with heights increasing from left to right. If Y also steps forward we have a row of h_X+1 boys commencing with Y with heights increasing.] Similarly if Y is taller than X k_Y must be greater than k_X.

Now as there are only n^2 different ordered pairs (h,k) where h,k are each one of 1, 2, 3, . . . , n, it follows that for some boy X either h_X or $k_Y > n+1$ Q.E.D.

332. In a number of years equal to the number of times a pig's mother is as old as the pig, the pig's father will be as many times as old as the pig as the pig is years old now. The pig's mother is twice as old as the pig will be when the pig's father is twice as old as the pig will be when the pig's mother is less by the difference in ages between the father and the mother than three times as old as the pig will be when the pig's father is one year less than twelve times as old as the pig is when the pig's mother is eight times the age of the pig.

When the pig is as old as the pig's mother will be when the difference in ages between the pig's father and the pig is less than the age of the pig's mother by twice the difference in ages between the pig's father and the pig's mother, the pig's mother will be five times as old as the pig will be when the pig's father is one year more than ten times as old as the pig is when the pig is less by four years than one-seventh of the combined ages of his father and mother. FIND THEIR RESPECTIVE AGES. (For the purposes of this problem, the pig may be considered to be immortal.)

Answer: Whew!! Possibly whoever composed this regarded it as the final reductio ad absurdum of this whole class of puzzle Let x,y,z represent the ages (in years) of the pig, its mother and its father respectively. The three sentences translate eventually into the equations (after simplification):

$$xz + y = x^3 + xy \quad (1)$$
$$-18x + 109y - 98x - 28|z-y| - 84 = 0 \quad (2)$$
$$-13x + 9y + 4z - 2|z-y| - 275 = 0 \quad (3)$$

Here $|z-y|$, the difference in ages between the pig's father and mother is equal to $z-y$ if $z > y$, but to $y-z$ if $y > z$.

Case 1. $z > y$. We obtain

$$-18x + 137y - 126z - 84 = 0 \quad (2)'$$

and

$$-13x + 11y + 2z - 275 = 0 \quad (3)'$$

which together with (1) can be solved to obtain $x = 3$, $y = 24$, $z = 25$. (After elimination of y and z one obtains $1660x^3 + 91x^2 - 3607x - 34{,}818 = 0$ whose only real solution is $x = 3$.)

Case 2. $z > y$. We obtain

$$-18x + 81y - 70z - 84 = 0 \quad (2)''$$
$$-13x + 7y + 6z - 275 = 0 \quad (3)''$$

which, with (1), yield $x \cong 3.055$, $y \cong 23.4$, $z \cong 25.0$. But these solutions are inconsistent with $z < y$. Hence no alternative answer has been found.

I suspect that we are expected to obtain a simpler solution of the puzzle by being aware of certain subtleties of the wording. The second sentence of the question concludes "twelve times as old as the pig *is* when the pig's mother is eight times the age of the pig". Can we conclude from the fact that the word in italics is neither "was" nor "will be" that the pig's mother is now 8 times as old as the pig? A similar "is" near the end of the third sentence might lead us to suspect also that the pig's age, x, is now 4 less than $(y+z)/7$.

Using either or both of

$$y = 8x \quad (4)$$

and

$$7x = y + z - 28 \quad (5)$$

in conjunction with the previous equations reduces the calculation of the solution marvellously. Similarly, if we are able to assume that x,y,z are all integers, the solution is greatly simplified.

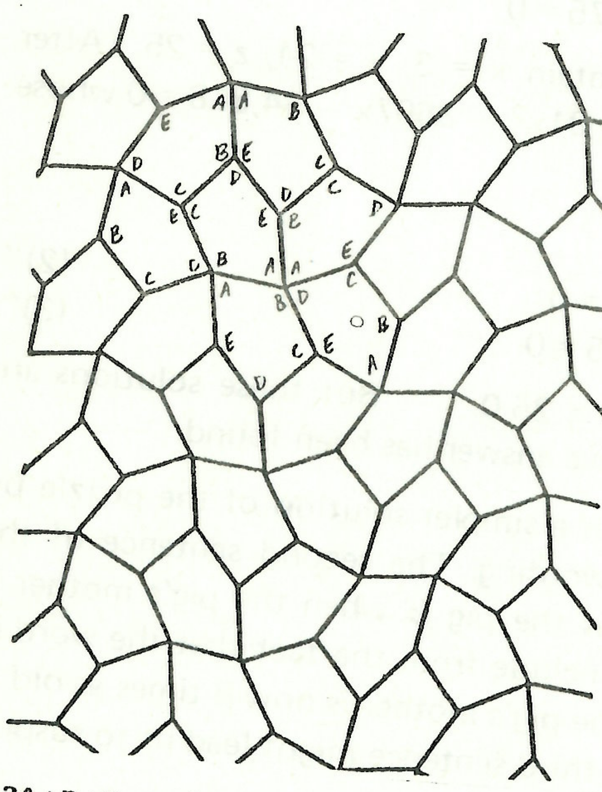
Successful Solvers of Problems 321–332

I. Doig (Sydney Grammar) 322, 324.
D. Dowe (Geelong Grammar) 321, 322, 323, 324, 325, 326, 327, 328, 329, 332.
J. O'Brien (St Joseph's) 321.
G. Reeves (Newington)321, 322, 323, 324.

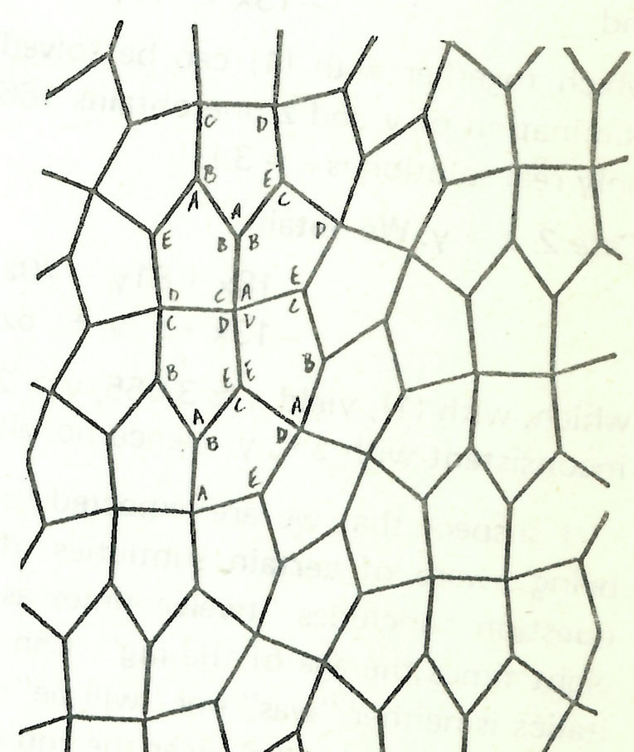
Late Solver

P. Bos (Sydney Boys' High) 317, 318.

☆ ☆ ☆



$2A+B+D=360^o$, $2C+E=360^o$, $B+D+E=360^o$
$A \simeq 73.38^o$, $B \simeq 128.49^o$, $C \simeq 106.62^o$,
$D \simeq 84.75^o$, $E \simeq 146.76^o$



$A+C+2D=360^o$, $A+2B=360^o$, $C+2E=360^o$
$A \cong 70.88^o$, $B \cong 144.56^o$, $C \cong 89.26^o$,
$D \cong 99.93^o$, $E \cong 135.37^o$

(See page 2)