MULTIGRADES

Everyone knows how to find examples of sets of three numbers which add up to the same number, such as

$$1 + 6 + 8 = 2 + 3 + 10$$

or sets of three numbers whose squares add up to the same number, such as

$$2^2 + 7^2 + 9^2 = 3^2 + 5^2 + 10^2$$

However, a more interesting example is two sets of three numbers which do both of these at the same time:

$$1 + 6 + 8 = 2 + 4 + 9$$
$$1^2 + 6^2 + 8^2 = 2^2 + 4^2 + 9^2.$$

A more interesting example still is the following two sets of four numbers:

$$1 + 5 + 8 + 12 = 2 + 3 + 10 + 11$$
$$1^2 + 5^2 + 8^2 + 12^2 = 2^2 + 3^2 + 10^2 + 11^2$$
$$1^3 + 5^3 + 8^3 + 12^3 = 2^3 + 3^3 + 10^3 + 11^3$$

or, more briefly,

$$1^n + 5^n + 8^n + 12^n = 2^n + 3^n + 10^n + 11^n \quad \text{for } n = 1,2,3.$$

This expression is called a multigrade of order 3, and the expression

$$1^n + 6^n + 8^n = 2^n + 4^n + 9^n \quad \text{for } n = 1,2$$

is called a multigrade of order 2. Similarly, you can think of multigrades of order 4, 5, 6, . . . , where the order of the multigrade is the largest value of n for which the equality holds. There are many methods of constructing multigrades of various orders and we shall look at a few in the course of this article.

One property of multigrades that is particularly interesting is that you can add the same integer to each term and still preserve the relationship. For example, we have:-

$$1^n + 6^n + 8^n = 2^n + 4^n + 9^n \quad \text{for } n = 1,2$$

By adding 6 to each term, we have:

$$7^n + 12^n + 14^n = 8^n + 10^n + 15^n \quad \text{for } n = 1,2.$$

The proof of this is easy as we can show for the case of a third order multigrade with 4 terms. Say we have:

$$A^n + B^n + C^n + D^n = E^n + F^n + G^n + H^n \quad \text{for } n = 1,2,3.$$

If we add k to each term, we have

$$(A+k) + (B+k) + (C+k) + (D+k) = (E+k) + (F+k) + (G+k) + (H+k).$$

Also

$$(A+k)^2 + (B+k)^2 + (C+k)^2 + (D+k)^2$$
$$= A^2 + B^2 + C^2 + D^2 + 2k(A+B+C+D) + 4k^2$$
$$= E^2 + F^2 + G^2 + H^2 + 2k(E+F+G+H) + 4k^2$$
$$= (E+k)^2 + (F+k)^2 + (G+k)^2 + (H+k)^2,$$

and

$$(A+k)^3 + (B+k)^3 + (C+k)^3 + (D+k)^3$$
$$= A^3 + B^3 + C^3 + D^3 + 3k(A^2+B^2+C^2+D^2) + 3k^2(A+B+C+D) + 4k^3$$
$$= E^3 + F^3 + G^3 + H^3 + 3k(E^2+F^2+G^2+H^2) + 3k^2(E+F+G+H) + 4k^3$$
$$= (E+k)^3 + (F+k)^3 + (G+k)^3 + (H+k)^3.$$

So much for that. Now let us have a look at a simple method of constructing a multigrade.

Start with a simple (first order) multigrade, such as:

$$1 + 4 = 2 + 3$$

Add 4 to each term

$$5 + 8 = 6 + 7.$$

By "switching sides" and combining, we obtain a longer multigrade.

$$1 + 4 + 6 + 7 = 2 + 3 + 5 + 8.$$

But, also

$$1^2 + 4^2 + 6^2 + 7^2 = 2^2 + 3^2 + 5^2 + 8^2$$

and so this is actually a second order multigrade.

The proof of this process is given later. When we added 4 to each term, we got a second order multigrade with all terms different and with 4 terms on both sides. By picking the value of k carefully, we can build up high order multigrades with reduced numbers of terms on each side. For example, to produce a third order multigrade, we start as before with a simple equality:

$$1 + 4 = 2 + 3$$

put $k = 3$

then $\quad 4 + 7 = 5 + 6.$

Hence $\quad 1^n + \cancel{4^n} + 5^n + 6^n = 2^n + 3^n + \cancel{4^n} + 7^n \quad \text{for } n = 1,2$

i.e. $\quad 1^n + 5^n + 6^n = 2^n + 3^n + 7^n \quad \text{for } n = 1,2$

put $k = 5$

$$6^n + 10^n + 11^n = 7^n + 8^n + 12^n \quad \text{for } n = 1,2$$

Switching and combining

$$1^n + 5^n + \cancel{6^n} + \cancel{7^n} + 8^n + 12^n = 2^n + 3^n + \cancel{6^n} + \cancel{7^n} + 10^n + 11^n \quad \text{for } n = 1,2,3$$

i.e. $\quad 1^n + 5^n + 8^n + 12^n = 2^n + 3^n + 10^n + 11^n \quad \text{for } n = 1,2,3$

We can continue in this way for as long as we wish to produce multigrades of any order we desire.

Let us now prove the "switching" procedure for second order multigrades. (Higher order "switching" can be similarly proved.) If we start with the simple equality

$$x + y = (x-z) + (y+z)$$

Adding k,

$$(x+k) + (y+k) = (x-z+k) + (y+z+k)$$

switching and combining

$$x + y + (x-z+k) + (y+z+k) = (x+k) + (y+k) + (x-z) + (y+z)$$

now take each side separately and sum the squares of the terms.

$$x^2 + y^2 + (x-z+k)^2 + (y+z+k)^2 = 2(x^2+y^2+z^2+k^2-xz+xk+yz+yk)$$
$$= (x+k)^2 + (y+k)^2 + (x-z)^2 + (y+z)^2.$$

This proves that the switching procedure yields a valid second order multigrade for all values of x,y,z and k.

Using properties of polynomials, it is possible to show that a multigrade of order n must have more than n terms on each side. This is obvious when $n = 1$, and you have been asked to prove it when $n = 2$ in the first problem below. This result led two mathematicians, Prouhet and Tarry, to propose the problem of generating multigrades of order n, with n+1 terms on each side of the equality. We have derived one solution above to this problem for order 3 and some other solutions for this order are given by the identity:

$$x^n + (y-z)^n + (3y+2z-2x)^n + (2y+3z-x)^n$$
$$= (2y-x)^n + (x-z)^n + (y+2z)^n + (3y+3z-2x)^n \quad \text{for } n = 1,2,3$$

where x,y and z are positive integers with $z \leqslant x \leqslant y$.

J.A.H. Hunter has derived a general identity for the third order problem, where the left hand terms with $n = 1$ form an arithmetic progression with common difference x^2+y^2:

$$0^n + (x^2+y^2)^n + (2x^2+2y^2)^n + (3x^2+3y^2)^n$$
$$= (2x^2-3xy+y^2)^n + (3y^2-xy)^n + (3x^2+xy)^n + (x^2+3xy+2y^2)^n, \quad \text{for } n = 1,2,3$$

where x,y are integers with $y \leqslant x \leqslant 3z$.

As an example, with $x = 3$ and $y = 2$, we have

$$0^n + 13^n + 26^n + 39^n = 4^n + 6^n + 33^n + 35^n \quad \text{for } n = 1,2,3$$

We then add any constant to get rid of 0^n (say 1):

$$1^n + 14^n + 27^n + 40^n = 5^n + 7^n + 34^n + 36^n \quad \text{for } n = 1,2,3$$

D.C. Cross has now derived a semi-general identity for the fifth order multigrades, and one of his solutions is

$$1^n + 9^n + 18^n + 38^n + 47^n + 55^n = 3^n + 5^n + 22^n + 34^n + 51^n + 53^n \quad \text{for } n = 1,2,3,4,5.$$

Problems

(1) If $x^n + y^n = a^n + b^n$ for $n = 1,2$, show that either $x = a$, $y = b$; or $x = b$, $y = a$. Thus a multigrade of order 2 must have at least 3 terms on each side.

(2) Use the switching procedure to get a general identity for a second order multigrade with three terms on each side.

(3) See if you can find some 3rd (or 4th) order multigrades, with 4 (or 5) terms on each side, which are not already given above.

(4) See if you can find another 5th order multigrade with 6 terms on each side.

B.J. Joyce

Mr Joyce is a chartered Civil Engineer living in Sydney whose hobby is recreational mathematics. I would like to thank him for the above article — Editor.