PROJECTILES

A projectile is a body projected by an external force which continues in motion for a certain time period under its own inertia.

Consider a projectile launched at an angle θ to the horizontal with an initial velocity u, as in Figure 1. In the figure, v_x represents the horizontal component of the projectile's velocity v, while v_y represents the vertical component and u_y the initial vertical velocity.

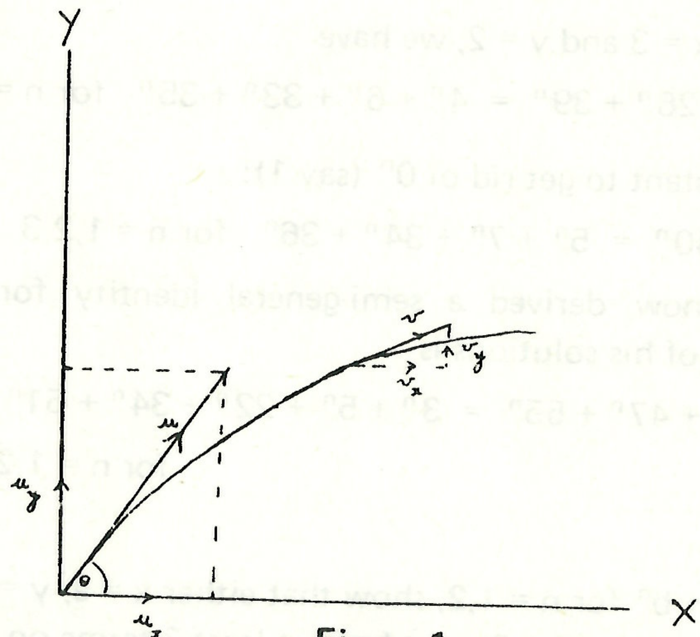

Figure 1

It is evident that the resolution of gravity through $\pi/2$ radians would give zero horizontal acceleration. Consequently the horizontal velocity component remains constant and is given by

$$v_x = u \cos \theta \qquad (1)$$

Thus the horizontal displacement x in time t is given by

$$x = u (\cos \theta)\, t \qquad (2)$$

The vertical motion of a projectile is unaffected by its horizontal motion. The projectile behaves vertically as would any linear moving object under the influence of gravity g. The vertical velocity component at the instant of projection is given by

$$u_y = u \sin \theta \tag{3}$$

Thus the vertical velocity component at any instant is given by

$$v_y = u \sin \theta - gt \tag{4}$$

Knowing the initial vertical velocity component we can write the vertical displacement y in time t as given by

$$y = u (\sin \theta)t - ½gt^2 \tag{5}$$

We can find the vertical displacement y for any horizontal displacement x simply by combining equations (2) and (5) to eliminate the parameter t, viz.,

$$y = x \tan \theta - gx^2 (\sec^2 \theta)/2u^2 \tag{6}$$

Now, since $\tan \theta$, $g \sec^2 \theta$ and $2u^2$ are all constant, equation (6) indicates that the path travelled by a projectile is parabolic. When the projectile is discharged straight up or along the ground the length of the parabola's latus rectum is zero or infinity respectively.

From Figure 1 one observes that at any instant the velocity of a projectile is given by

$$v^2 = v_x{}^2 + v_y{}^2 \tag{7}$$

If we replace v_x by $u \cos \theta$, from (1), and v_y by $u \sin \theta - gt$, from (4), we get

$$\begin{aligned} v^2 &= u^2 \cos^2 \theta + u^2 \sin^2 \theta - 2u (\sin \theta) gt + g^2 t^2 \\ &= u^2 (\cos^2 \theta + \sin^2 \theta) - 2g(u \sin \theta t - ½gt^2) \\ &= u^2 - 2gy \end{aligned} \tag{8}$$

where this velocity vector makes an angle with the horizontal given by

$$a = \tan^{-1} (v_y / v_x) \tag{9}$$

The greatest height attained by the projectile occurs when the projectile stops rising: which is when the vertical velocity component becomes zero. Thus, if we replace v_y by zero in equation (4) we get time taken to reach maximum height is given by

$$t_M = u(\sin \theta)/g = u_y / g \tag{10}$$

Therefore total time of flight is given by

$$t_T = 2t_M = 2u_y / g \tag{11}$$

because the parabola is symmetrical.

To find the maximum height we substitute t_M in equation (5). Therefore we find the maximum height attained by the parabola is given by

$$y_M = (u^2 \sin^2\theta)/2g = u_y{}^2/2g \qquad (12)$$

To find the horizontal range we substitute t_T in equation (2).
Therefore we find the horizontal range is given by

$$\text{Range} = (2u^2 \sin\theta \cos\theta)/g$$
$$= (u^2 \sin 2\theta)/g \qquad (13)$$

The factor $\sin 2\theta$ has a maximum value of 1 which occurs when $2\theta = \pi/2$, and so $\theta = \pi/4$. Thus to achieve maximum range the projectile must be launched at an angle to the horizontal given by

$$\theta = \pi/4 \qquad (14)$$

whence the maximum horizontal range will be

$$\text{Maximum range} = u^2/g \qquad (15)$$

I will now show how a projectile obeys the law of conservation of energy. The two forms of energy with which we are concerned are kinetic energy and potential energy. The kinetic energy of the projectile with velocity v is given by

$$\text{K.E.} = \tfrac{1}{2}mv^2 \qquad (16)$$

where m is the mass of the projectile. The kinetic energy of course exists in two components, the horizontal and vertical. Hence we can replace v^2 in equation (16) by u^2-2gy from equation (8) which gives us the total kinetic energy of the projectile as being

$$\text{K.E.} = \tfrac{1}{2}m(u^2-2gy) \qquad (17)$$

The potential energy of the projectile is given by

$$\text{P.E.} = mgy$$

where m is the mass. The potential energy exists only as mgy. The total energy of the projectile is the sum of its kinetic and potential energy. Therefore we get the total energy of the projectile as given by

$$E = \tfrac{1}{2}m(u^2-2gy) + mgy$$

i.e.

$$E = \tfrac{1}{2}mu^2 \qquad (18)$$

which is constant for all positions of the projectile on its path.

In all the above studies we have neglected air resistance. Let us now consider this effect. Air resistance is a force proportional to the velocity of the projectile.

The air acts in the direction opposite to that in which the projectile is moving. Now since horizontal velocity and vertical velocity both vary then so too does horizontal and vertical resistance. The net resistance on the projectile can be determined by vectorially adding these components.

Consider the vertical component of the projectile's motion. Since the vertical resistance R_y to the motion varies as the velocity v_y then

$$R_y = k_1 v_y \tag{19}$$

where k_1 is a constant. For simplification let $k_1 = mk$ and $g' = g/k$. So we have

$$R_y = mkv_y \tag{20}$$

For the upward journey the force acting on the projectile in the downward direction is given by

$$F_D = mg + R_y = mk(g' + v_y) \tag{21}$$

Now by Newton's second law, the force $-F_D$ acting on the projectile in the upward direction is equal to

$$F_U = m\,dv_y/dt \tag{22}$$

Thus

$$m\,dv_y/dt = -mk(g' + v_y)$$

$$dv_y/dt = -k(g' + v_y)$$

so

$$dt/dv_y = -1/k(g' + v_y)$$

Hence $kt = -ln(kg' + kv_y) + c_1$ where c_1 is a constant. When $t = 0$, $v_y = u \sin\theta$, and so we see

$$c_1 = ln(kg' + ku \sin\theta).$$

Thus

$$kt = ln(kg' + ku \sin\theta) - ln(kg' + kv_y),$$

which yields the time equation for the upward journey t_U as

$$kt_U = ln[g' + u \sin\theta)/(g' + v_y)] \tag{23}$$

At the highest point, $v_y = 0$. Therefore time t_M taken to reach maximum height is given by

$$kt_M = ln(1 + (u \sin\theta)/g') \tag{24}$$

We now wish to find the maximum height. Using the fact that $dv_y/dt = -k(g' + v_y) = -k(g' + dy/dt)$, we get

$$v_y = -k(g't + y) + c_2$$

where c_2 is a constant. When $t = 0$, $y = 0$ and $v_y = u \sin \theta$. Therefore we see

$$c_2 = u \sin \theta$$

which yields the vertical velocity equation for the upward journey $v_{y,U}$ as

$$v_{y,U} = -k(g't + y) + u \sin \theta \tag{25}$$

Rearranging equation (25) gives us the equation for vertical displacement y_U for the upward journey as

$$ky_U = -kg't - v_y + u \sin \theta \tag{26}$$

substituting the value for t_U given in equation (23) we get

$$ky_U = u \sin \theta - v_y - g' \ln \left(\frac{g' + u \sin \theta}{g' + v_y} \right) \tag{27}$$

At the highest point on the path, $v_y = 0$. Therefore, the maximum height is given by

$$ky_M = u \sin \theta - g' \ln(1 + (u \sin \theta)/g') \tag{28}$$

For the downward journey, let y be the vertical displacement *downwards* from the highest point. The force on the projectile in that direction is

$$F_D = mg - R_y = mk(g' - v_y) \tag{29}$$

Once again by Newton's second law the force acting on the projectile in the downward direction is equal to

$$F_D = m dv_y/dt \tag{30}$$

Thus

$$m dv_y/dt = mk(g' - v_y)$$

i.e.

$$dv_y/dt = k(g' - v_y)$$

$$dt/dv_y = 1/k(g' - v_y)$$

Hence

$$kt = -\ln(kg' - kv_y) + c_3$$

where c_3 is a constant. When $t = 0$, $v_y = 0$. Therefore we see

$$c_3 = \ln kg'$$

Thus

$$kt = -\ln(kg' - kv_y) + \ln kg'$$

which yields the time equation for the downward journey (t_D) as

$$kt_D = -\ln(1 - v_y/g') \tag{31}$$

Using the fact that

$$dv_y/dt = k(g'-v_y) = k(g'-dy/dt),$$

we get

$$v_y = k(g't-y) + c_4$$

where c_4 is a constant. When $t = 0$, $y = 0$ and $v_y = 0$. Therefore we see

$$c_4 = 0$$

which yields the equation for vertical velocity for the downward journey $v_{y,D}$ as

$$v_{y,D} = k(g't-y) \tag{32}$$

Rearranging equation (32) gives us the equation for vertical displacement y_D for the downward journey as

$$ky_D = kg't-v_y \tag{33}$$

Substituting the value for t given in equation (31), we get

$$ky_D = -g' \ln(1-v_y/g') - v_y \tag{34}$$

We now wish to find the time taken to fall. Consider the projectile when it reaches the plane from which it was projected. It will have fallen a distance given by equation (28) from its highest point in a time (say t_F) and will strike the ground with a velocity (w say). Therefore from equation (32) we get

$$w = k(g't_F - y_M) \tag{35}$$

Now, substituting w into equation (31), we get, when the projectile reaches the plane again,

$$kt_F = -\ln(1-w/g') \tag{36}$$

Combining equations (35) and (36) to eliminate the parameter w, we get

$$kt_F = -\ln(1-kt_F + ky_M/g') \tag{37}$$

Writing $\exp(-kt_F)$ for the exponential e^{-kt_F}, we get

$$\exp(-kt_F) = 1-kt_F + ky_M/g' \tag{38}$$

If we substitute the value for y_M given in equation (28) we get time taken to fall from maximum height, t_F is the solution of the following

$$\exp(-kt_F) + kt_F = 1 + (u \sin\theta)/g' - \ln[1 + (u \sin\theta)/g'] \tag{39}$$

which is a very difficult equation to solve. Perhaps someone may like to try and if successful write to Parabola.

From equation (31) we get

$$v_y = g' - g' \exp(-kt_D) = g/k - g \exp(-kt_D)/k \quad (40)$$

From equation (40) we see that v_y is always less than g/k: a fact which is small comfort to a man falling out of a plane (where k is very small), but useful if he is wearing a parachute (where k is larger). From this we also see the terminal velocity of a falling body is given by

$$v_\infty = g/k \quad (41)$$

because as the time t_D gets larger and larger, the exponential term $\exp(-kt_D)$ becomes more and more negligible.

Consider the horizontal component of the projectile's motion. Since the horizontal resistance R_x to the motion varies as the velocity v_x then

$$R_x = k_2 v_x \quad (42)$$

where k_2 is a constant. For simplification let $k_2 = mk'$ and so we have

$$R_x = mk'v_x \quad (43)$$

The horizontal component of the force acting on the projectile is simply R_x as gravity has no horizontal component. Now the air resistance acts in the direction opposite to that in which the projectile is moving. So by Newton's second law,

$$-R_x = m\, dv_x/dt \quad (44)$$

Thus

$$m\, dv_x/dt = -mk'v_x$$

i.e.

$$dv_x/dt = -k'v_x$$

$$dt/dv_x = -1/k'v_x$$

Hence

$$k't = -\ln v_x + c_5$$

where c_5 is a constant. When $t = 0$, $v_x = u \cos\theta$. Therefore we see

$$c_5 = \ln(u \cos\theta).$$

Thus the time equation for the horizontal component is given by

$$k't = -\ln v_x + \ln(u\cos\theta) = -\ln[v_x/(u\cos\theta)] \quad (45)$$

Rearranging equation (45) gives us the equation for horizontal velocity as

$$v_x = u\cos\theta . e^{-k't} \quad (46)$$

This result shows that the parachute used to slow down jet cars or flaps used on aeroplanes do not stop the car or plane in a finite time. This is true because $e^{-k't}$ is always greater than zero no matter how small it is and hence the projectile always has some horizontal velocity.

Rewriting equation (46) as

$$dx/dt = (u \cos\theta)\, e^{-k't}$$

we get when integrating, that

$$k'x = -(u \cos\theta)\, e^{-k't} + c_6$$

where c_6 is a constant. When $t = 0$, $x = 0$. Therefore we see

$$c_6 = u \cos\theta .$$

Thus the displacement x in time t is given by

$$k'x = -(u \cos\theta)\, e^{-k't} + u \cos\theta$$

i.e.

$$= u \cos\theta\, (1 - e^{-k't}) \tag{47}$$

This final equation shows that a particle can never go further than $(u \cos\theta)/k'$ units along the ground even on a perfectly smooth surface.

Philip Bos, Sydney High

This is a good article by Philip. Maybe you might like to contribute an article? — Editor.

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RESEARCH CORNER

As the last issue of Parabola was so late, there have been no answers to our questions on Palindromic numbers. To remind you, a palindromic number is a positive integer which reads the same backwards as forwards, and the questions may be found in Vol. 13 No. 1.

Since we asked the questions, Sanyo have very kindly offered to donate a pocket calculator which will be given to the student with the best answers to these questions. So start researching!