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BASIC AREAS AND VOLUMES

Gavin Brown*

1. Why?

Here is a very pretty argument for finding the area of a circle:

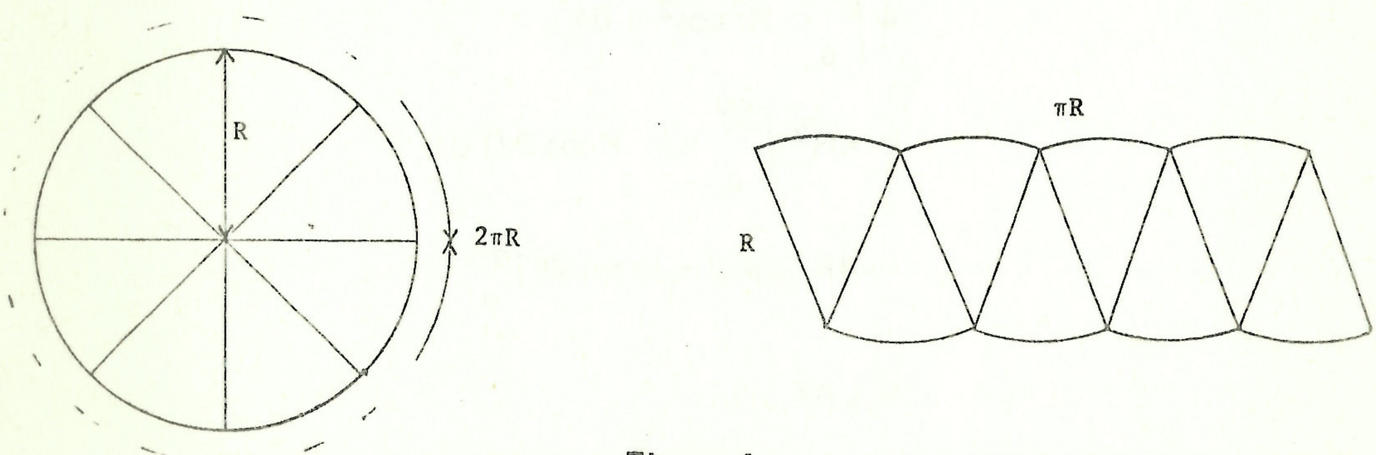

Figure 1

The idea is to chop a circle of radius R into a large even number of congruent segments. Then we reassemble the segments to look like a rectangle. If we take enough segments then the new shape will get closer to a rectangle whose height is approximately R and whose length is half the circumference. Since the area of a rectangle is the height times the length we come up with R times πR, in other words πR^2.

That argument is very enjoyable and quite plausible but it's not really very sound as it stands. Because it is visually compelling this is an excellent heurism for use in early grades but it is not often made precise. It is, of course, possible to tidy it up so that it becomes a proof.

* *Professor Brown is the Head of the Department of Pure Mathematics at the University of N.S.W.*

Now here is a very **nasty** argument for finding the area of a circle: (Don't let it put you off the rest of this article, my intention is to show a much nicer way later!)

$$f(x) = \sqrt{R^2-x^2}$$

The area of the circle is

$$4\int_0^R \sqrt{R^2-x^2}\,dx.$$

Let $x = R\sin\theta$, so that the integral becomes

$$4\int_0^{\pi/2} R^2\cos^2\theta\,d\theta$$

$$= 4R^2\int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 4R^2\left[\theta/2 + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \pi R^2.$$

I don't like this argument at all! It is unnecessarily sophisticated in many ways and uses several technical results.

The trouble is that many calculus textbooks use this argument or something like it to find the area of a circle — or worse still they spend ages developing a theory of integration but simply assume that everybody knows the area of a circle. Of course everybody does — but usually they have only met a clever plausibility argument like the first one I gave. That's the reason for this article. I want to make sure there is something between the two extremes and show the simplest ways I know to derive basic areas and volumes.

2. What we use

For a start we shall assume that the circumference of a circle of radius R is $2\pi R$. This isn't really much of an assumption because it is virtually the definition of π! (To be accurate we would define 2π as the circumference of a circle of radius one unit and prove that, in general, the circumference of a circle is proportional to the radius.)

Next we suppose that we know how to differentiate functions like x^3. That certainly isn't much of a demand!

Finally we assume that a function whose derivative is everywhere zero must be constant. When it comes to applying mathematics, this result can safely be assumed, I believe. The point is that it is rather less of an assumption than the other assumptions usually involved in accepting that mathematical models fit reality! In particular we must accept this result if we want to apply calculus to dynamics because in that context it translates into the statement that a particle whose instantaneous velocity is always zero remains stationary! In other words we intend to assume that a particle which is "at rest" does not move!

3. Circles within circles

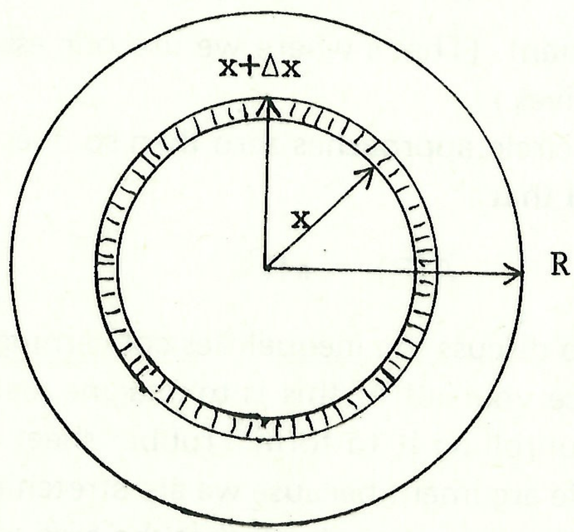

Figure 2

Think of a disc of radius R being decomposed as a collection of concentric rings. Let $A(x)$ denote the area of a circle of radius x and suppose $\Delta x > 0$. Then $A(x + \Delta x) - A(x)$ is the area of the ring which has inner radius x and outer radius $x + \Delta x$. The area of this ring lies between $2\pi x \Delta x$ and $2\pi(x + \Delta x)\Delta x$. (Unlike the first argument in section 1 this does not involve any approximation and there is no need to assume that Δx is small. We will return to this point later.) Now we have

$$2\pi x \leqslant \frac{A(x + \Delta x) - A(x)}{\Delta x} \leqslant 2\pi(x + \Delta x).$$

Let $\Delta x \to 0$ and note that both "ends" of the inequality tend to $2\pi x$. It follows that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{A(x+\Delta x) - A(x)}{\Delta x} = 2\pi x.$$

If we feel that way inclined, we can run through the corresponding argument with $\Delta x < 0$. However that's not really needed, because $A(x)$ is certainly going to be a smooth function of x, so we now know that

$$\frac{dA(x)}{dx} = 2\pi x.$$

Of course the function $A(x) = \pi x^2$ satisfies this equation, therefore the most general possible solution is

$$A(x) = \pi x^2 + C,$$

where C is some constant. (That's where we use our assumption about functions with vanishing derivatives.)

As the radius of a circle approaches zero then so does the area of that circle. It follows that $C = 0$ and that

$$A(R) = \pi R^2.$$

We should return to discuss the inequalities concerning the area of the ring. The quick way to convince yourself of this is to imagine making a Swiss roll! Imagine cutting the ring and unrolling it to form a rubber sheet of thickness Δx. However this is only a plausible argument because we are stretching the ring in the process. To find a precise argument we can check that the area of the ring must exceed Δx x the perimeter of any polygon inscribed in the circle of radius x, and that the area is less than Δx x the perimeter of any polygon circumscribing the circle of radius $x + \Delta x$.

I have been rather fussy about writing down these arguments because I wanted to emphasize that every step can be made accurate and can be reduced to primitive assumptions. After one gets the hang of this kind of argument one can scribble down a rough sketch of the idea along the following lines:

"A circle is made up of concentric rings of area approximately $2\pi x\, dx$. Hence the area is

$$\int_0^R 2\pi x \, dx = \pi R^2. \quad "$$

Of course the first sentence of that sketch is inaccurate (indeed it is technically meaningless) but even pure mathematicians write that sort of thing consciously all the time (at least this pure mathematician does!). In my own research work I find it very useful to rough out proofs without bothering to get the details correct. Then if it seems to be leading somewhere I come back and check the details to make sure the argument stands up. Sometimes it does.

4. Slicing spheres

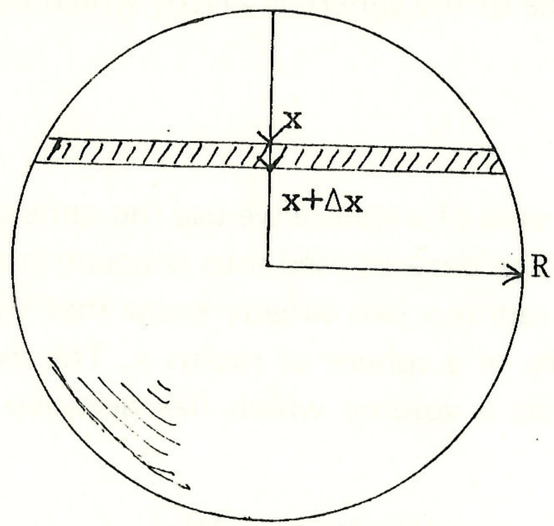

Figure 3

It's now pretty easy to find the volume of a sphere by slicing with horizontal cuts. In fact the volume lying between the depths x and $x + \Delta x$ must lie between the volume of a circular cylinder of radius $\sqrt{R^2-(R-x)^2}$ and height Δx and that of a circular cylinder of radius $\sqrt{R^2-(R-x-\Delta x)^2}$ and height Δx. It is no great strain to prove that the volume of a cylinder is the area of the base times the height (we can approximate the volume by using lots of parallelepipeds of the same height but small base areas) so we come up with another useful inequality,

$$\pi(R^2 - (R-x)^2)\,\Delta x \leqslant H(x+\Delta x) - H(x) \leqslant \pi(R^2 - (R-x-\Delta x)^2)\,\Delta x,$$

where $H(x)$ denotes the volume of the "hemispherical" cup of depth x cut from a sphere of radius R, and Δx is positive.

Now we find that

$$\lim_{\Delta x \to 0} \frac{H(x+\Delta x) - H(x)}{\Delta x} = \pi(2xR - x^2).$$

This means we obtain the equation

$$\frac{dH}{dx} = \pi(2xR - x^2),$$

and deduce that

$$H(x) = \pi(Rx^2 - x^3/3).$$

In particular the volume of the sphere is $2H(R)$ which equals $4\pi R^3/3$.

5. Concentric shells

To find the surface area of a sphere we use the same diagram as in section 3 but now it represents a sphere decomposed into concentric shells. Let $V(x)$ denote the volume of a sphere of radius x (we already know that $V(x) = 4\pi x^3/3$) and let $S(x)$ denote the surface area of a sphere of radius x. The shell with inner radius x and outer radius $x + \Delta x$ has a volume which lies between $S(x)\Delta x$ and $S(x + \Delta x)\Delta x$. Thus we find that,

$$S(x) \leqslant \frac{V(x + \Delta x) - V(x)}{\Delta x} \leqslant S(x + \Delta x).$$

However $S(x)$ is certainly a smooth function of x so we know that $S(x + \Delta x)$ tends to $S(x)$ as Δx tends to zero. Accordingly the inequality yields the following equation, when we let $\Delta x \to 0$,

$$S(x) = \frac{dV(x)}{dx}.$$

We already know that $V(x) = 4\pi x^3/3$. It follows that $S(x) = 4\pi x^2$, and the surface area of a sphere of radius R is $4\pi R^2$.

6. Chopping cones

We can find the volume of a right circular cone with base radius R and height H by the slicing process used in section 4. (This can also be done by proving that the volume of a pyramid is 1/3 base x height and approximating by slanted pyramids. That seems a bit harder though.)

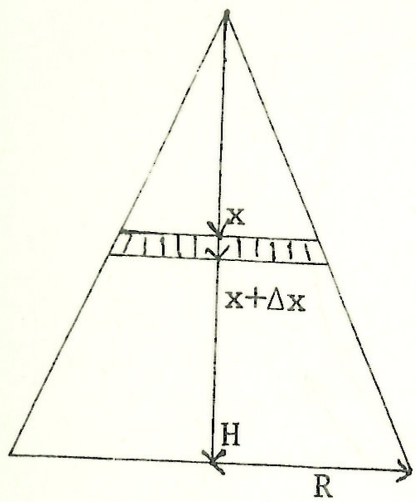

Figure 4

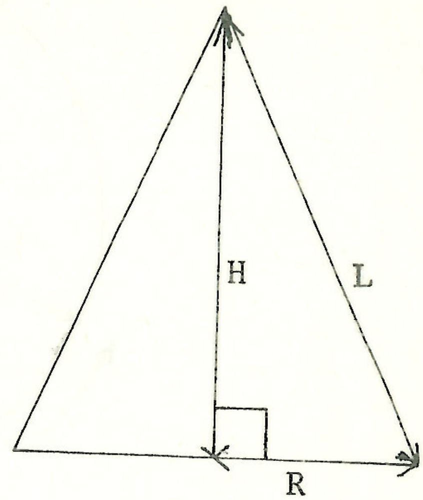

Figure 5

Using similar triangles we see that the cross-sectional radius at depth x is Rx/H. Accordingly we see that

$$\pi R^2 x^2 \Delta x/H^2 \leqslant C(x+\Delta x) - C(x) \leqslant \pi R^2 (x+\Delta x)^2 \Delta x/H^2,$$

where $C(x)$ is the volume to depth x.

It follows that

$$\pi R^2 x^2 = H^2 \frac{dC}{dx},$$

and hence that

$$C(x) = \pi R^2 x^3/3H^2.$$

The volume of the cone is therefore $C(H) = \pi R^2 H/3$.

In order to write down a neat formula for the surface area of a right circular cone it is convenient to introduce the "slant height" L. (Of course $L = \sqrt{R^2 + H^2}$. See figure 5 above.)

The surface area of the curved surface is πRL. My favourite way of seeing this involves cutting down one "edge" and rolling the surface out to form a sector of a circle of radius L.

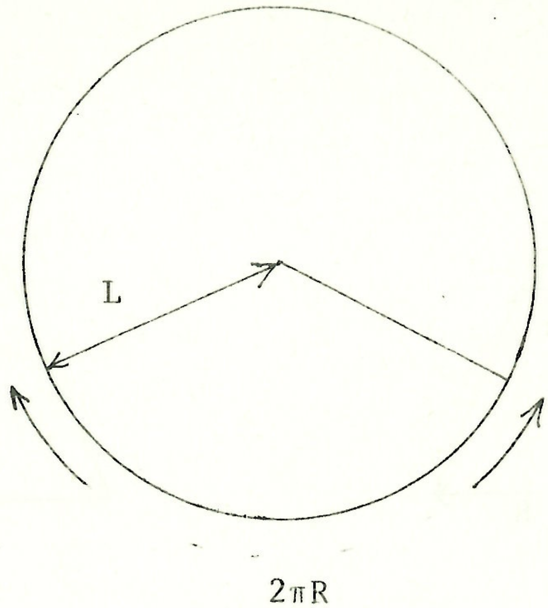

Figure 6

The area of this sector bears the same proportion to the area of the circle of radius L as does the arc of the sector to the circumference of that circle. Hence the required conical surface area, K, must satisfy

$$K/\pi L^2 = 2\pi R/2\pi L.$$

In other words,

$$K = \pi R L.$$

7. Cylindrical sleeves and skull caps

One of the prettiest results about surface areas is as follows:

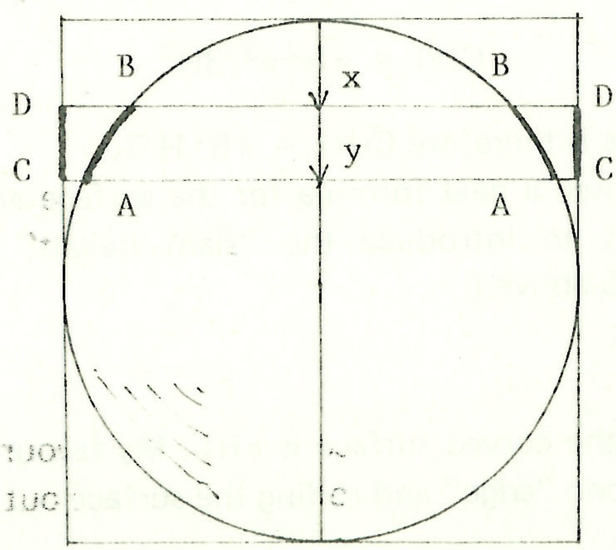

Figure 7

If a sphere is cut by two parallel planes then the surface area of the enclosed portion is the same as the surface area of the enclosed portion of the circumscribing cylinder whose base is parallel to the planes. (The area obtained by rotating AB about the central vertical axis is the same as that obtained by rotating CD about that axis.)

Any calculus proof of this I have seen makes very heavy weather of it indeed. Here is my version which avoids any nasty integrals.

The main step is to find the surface area of the cap of depth x.

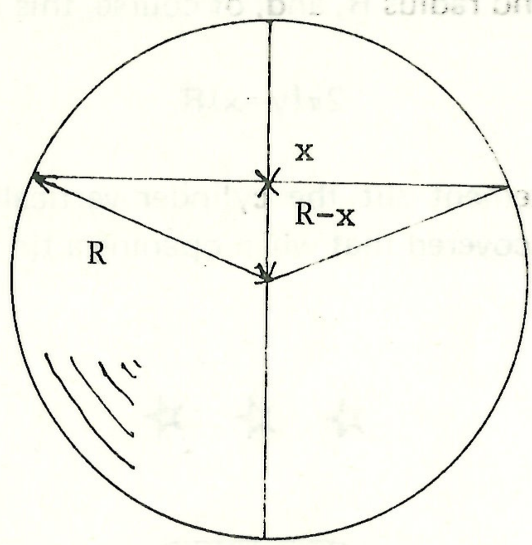

Figure 8

The surface area of this cap bears the same proportion to the surface area of the sphere as does the volume of the solid "sector" it subtends to the volume of the sphere. That "sector" is made up of the solid cap together with a right circular cone of height R−x and base radius $\sqrt{R^2-(R-x)^2}$. Using the results of sections 4 and 6, we find that the solid sector has volume

$$\pi(Rx^2 - x^3/3) + \frac{1}{3}\pi(R^2 - (R-x)^2)(R-x)$$

which simplifies to

$$\frac{2}{3}\pi x R^2.$$

Hence the surface area of our cap must be

$$\frac{2\pi x R^2/3}{4\pi R^3/3} \times 4\pi R^2 = 2\pi x R.$$

Now look back at Figure 7. Suppose that the one plane is at depth x and that the other plane is at depth y. The surface area of the part of the sphere between these planes is the difference between the surface area of the corresponding caps and hence is

$$2\pi(y-x)R.$$

The surface area of the relevant part of the circumscribing cylinder is that of a cylinder of height y–x and radius R, and, of course, this is again

$$2\pi(y-x)R.$$

(To prove the last statement slit the cylinder vertically and unroll to form a rectangle. Pythagoras discovered that when opening a tin of beans.)

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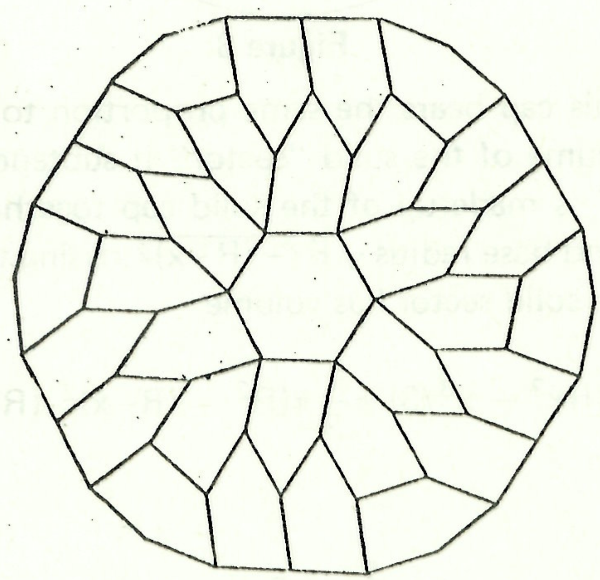

Tiling found by J. Hirschhorn*

**Jeremy is in Kindergarten at Christ the King's School, North Rocks.*