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# A FORMULA FOR PRIME NUMBERS, PART II

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In answer to my article in Vol. 12 No. 3, the editor has asked if it is always possible to select numbers A and B to satisfy the conditions of the formula. I believe that I am in a position to prove that these numbers can be chosen.

I define two terms as follows: let P(n) denote the n'th prime number (thus P(1) = 2, P(2) = 3 and so on) and let

$$N(n) = P(1) \times P(2) \times \ldots \times P(n).$$

Now choose A = N(n). Then A–B is prime if

(i) A and B have no common factor greater than 1,

and (ii) $A - P(n)^2 < B < A - 1$.

Thus, if such a B exists, it is not divisible by any of P(1), P(2), . . . , P(n), and it is among the $P(n)^2 - 2$ consecutive integers

$$A - P(n)^2 + 1,\ A - P(n)^2 + 2, \ldots, A-2.$$

Now, in answer to my letter in Vol. 13 No. 1, the editor has pointed out that the number of numbers not divisible by any of P(1), P(2), . . . P(n) among the first N integers is approximately equal to

$$N(1-1/P(1))(1-1/P(2)) \ldots (1-1/P(n)).$$

A little thought shows that it does not have to be the first N consecutive positive integers. In any set of N consecutive positive integers, approximately N/P(1) are divisble by P(1), N/P(2) by P(2) and so on, and the result follows as the editor showed in Vol. 13 No. 1.

So, the number of suitable values of B is given approximately by

$(P(n)^2 - 2)(1 - 1/P(1))(1 - 1/P(2)) \ldots (1 - 1/P(n))$, which we shall call F(n).

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The following table shows how well F(n) approximates the actual number of suitable B for small values of n:

| n | A | Suitable values for B | F(n) |
|---|---|---|---|
| 3 | 30 | 7, 11, 13, 17, 19, 23 | 6.1 |
| 4 | 210 | 163, 167, 169, 173, 174, 181, 187, 191, 193, 197, 199 | 10.3 |

Clearly, the error caused by using F(n) to approximate the actual number of suitable B is small. So, in order to prove that there is at least one suitable value for B, it need only be shown that F(n) increases as n increases. To do this, consider F(n)/F(n–1).

$$\begin{aligned} F(n)/F(n-1) &= (1 - 1/P(n))(P(n)^2 - 2)/(P(n-1)^2 - 2) \\ &> (1 - 1/P(n))\,P(n)^2/P(n-1)^2 \\ &= P(n)(P(n) - 1)/P(n-1)^2 \\ &> 1. \end{aligned}$$

So to sum up, I have shown that as n increases, F(n) does also. I have shown that, since the error is small, there will always be a value of B satisfying the stipulated conditions. And this all goes to prove that it is always possible to find values for A and B to satisfy the conditions of the formula.

**Editor's comment:** This is an excellent article. Philip has gone a long way towards answering the question raised concerning his formula. He has given us good reason to believe that the numbers A and B can be found. However, he has not given a proof in the accepted sense. (Just because a statement is true for small values of n, it isn't necessarily true for all values of n.)

However, if we do as Philip does, and choose A = N(n), then it is a fact that there are as many suitable choices for B as there are prime numbers between P(n) and $P(n)^2$ (each value of B yields one of these primes, and each of these primes is given by the formula A–B for some B satisfying the conditions). So the formula yields primes if and only if there are primes between P(n) and $P(n)^2$. And it is known that there are such primes. So Philip's formula works!

You may be interested to learn that $P(n+1) < 2P(n)$; in other words, each prime is less than twice the previous prime. An equivalent statement is that for every $x \geq 2$, there is a prime between x and 2x. Can you prove that these statements are equivalent?