PROBLEM SECTION

You are invited to submit solutions to one or more of these problems. Answers should bear your name, year, and school. Solutions will be published in the issue after next.

The last five of these problems appeared in the 1977 International Mathematical Olympiad, one of the most famous school mathematics competitions in the world. It is conducted in two 4-hour sessions, three questions being presented to the candidates in each session. Several of the other problems appeared some years ago in an American competition, the Wisconsin Mathematical Engineering and Science Talent Search.

357. Chess-players from two schools competed. Each player played one game with every other player. There were 66 games among players from one school, and in all there were 136 games. How many players from each school entered the tournament?

358. What are the last two digits of $2^{2^{73}}$? Show your working.

359. An infinitely long list is made of all the pairs of integers m,n for which $23m-10n$ is exactly divisible by 17. Another list is made of all the pairs for which $7x + 11y$ is exactly divisible by 17. Prove that the two lists are exactly like.

360. Suppose $a(1), a(2), \ldots, a(k)$ and $b(1), b(2), \ldots, b(k)$ are integers such that $a(1) \geqslant b(1) \geqslant 1$, $a(2) \geqslant b(2) \geqslant 1$, and so on. Let $a = a(1) + a(2) + \ldots a(k)$, and $b = b(1) + b(2) + \ldots + b(k)$.

(i) Prove that the product

$$[b(1)(a(1)-b(1)) + 1]\ [b(2)(a(2)-b(2)) + 1] \times \ldots \times [b(k)(a(k)-b(k)) + 1]$$

is greater than or equal to $a-b+1$.

(ii) Can you determine exactly under what conditions equality occurs?

361. A number of blocks, each 2cm x 2cm x 1cm, have been fitted snugly together to make a solid 20 cm high. (The top dimensions of the solid are, say, m cm by n cm.) A straight line, parallel to the 20 cm sides pierces the solid from top to bottom. Prove that the straight line cannot pierce exactly one of the blocks.

362. (i) In the morning a working man leaves his cat in the house. The house has one door which has been left open. When the man returns in the evening the cat is outside. Prove that the cat crossed the threshold an odd number of times.

(ii) A triangle ABC is the union of a finite family, F, of triangles. If two different triangles in F intersect, they intersect in a vertex of both or an edge of both. Colour each of the vertices of the triangles in F red, blue or yellow. Colour A red, B blue, and C yellow. If a vertex V lies on AB, colour it red or blue, if V lies on BC, colour it blue or yellow, if V lies on CA, colour it red or yellow. Prove that the number of triangles in F which have one red, one blue and one yellow vertex is odd.

363. An absent-minded bank-clerk switched the dollars and cents when he cashed a cheque for Mr Brown, giving him dollars instead of cents and cents instead of dollars. After buying a five-cent newspaper Mr Brown discovered that he had left exactly twice as much as his original cheque. What was the amount of his cheque?

364. Four points K,L,M,N inside a square ABCD are such that ABK, BCL, CDM and DAN are all equilateral. Prove that the mid-points of the line segments KL, LM, MN, NK, AK, BK, BL, CL, CM, DM, DN, and AN are the vertices of a regular 12 sided polygon.

365. a(1), a(2), a(3), . . ., a(N) is a list of real numbers such that the sum of every seven consecutive numbers in the list is negative but the sum of every 11 consecutive numbers is positive.

Find the maximum length of the list (i.e. the largest possible value of N) and give an example of such a list of maximum length.

366. For any natural number n greater than 2 denote by V(n) the collection of all numbers expressible in the form 1 + kn where k is a positive integer. A number x in V(n) will be called indecomposable if there do not exist elements y,z in V(n) such that x = yz (e.g. 25 is indecomposable in V(3)). Show that V(n) contains numbers which can be factorised in two different ways into indecomposable factors. (i.e. it is possible to have uv = wx in V(n) where all of u,v,w and x are indecomposable but $u \neq w$, $u \neq x$).

367. Let a and b be positive integers such that, when $a^2 + b^2$ is divided by a + b the quotient q and the remainder r satisfy the equation $q^2 + r = 1977$. Find all possible values of a and b.

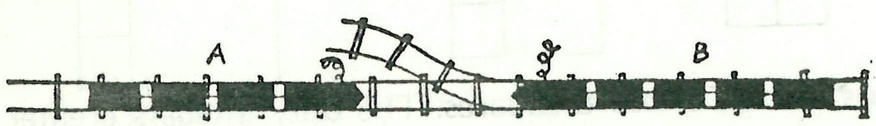
368. The infinite sequence

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

consists of positive integers. (i.e. f(n) is always a positive integer) and satisfies the inequality $f(n+1) > f(f(n))$ for every positive integer n. Prove that $f(n) = n$ for every n.

Solutions to Problems 333-344 (Vol. 13 No. 1)

333. Two trains A and B are travelling in opposite directions on a line with a single track and wish to pass with the help of a siding (see figure).



The siding will only take one carriage or one engine at a time and can only be entered from the right. If train A has 3 carriages and train B has 4 carriages, how can they pass with the minimum number of moves?

Solution by M. Garth (Castle Hill High), slightly condensed.

Assuming that a carriage cannot be pushed by hand, and that carriages can be hitched to the front of an engine, this is the minimal solution. The carriages are labelled A1, A2, A3 and B1, B2, B3, B4. A move consists of any part or all of the train(s) moving in one direction.