366. For any natural number n greater than 2 denote by V(n) the collection of all numbers expressible in the form 1 + kn where k is a positive integer. A number x in V(n) will be called indecomposable if there do not exist elements y,z in V(n) such that x = yz (e.g. 25 is indecomposable in V(3)). Show that V(n) contains numbers which can be factorised in two different ways into indecomposable factors. (i.e. it is possible to have uv = wx in V(n) where all of u,v,w and x are indecomposable but $u \neq w$, $u \neq x$).

367. Let a and b be positive integers such that, when $a^2 + b^2$ is divided by a + b the quotient q and the remainder r satisfy the equation $q^2 + r = 1977$. Find all possible values of a and b.

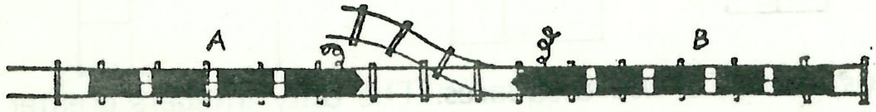
368. The infinite sequence

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

consists of positive integers. (i.e. f(n) is always a positive integer) and satisfies the inequality $f(n+1) > f(f(n))$ for every positive integer n. Prove that f(n) = n for every n.

Solutions to Problems 333-344 (Vol. 13 No. 1)

333. Two trains A and B are travelling in opposite directions on a line with a single track and wish to pass with the help of a siding (see figure).



The siding will only take one carriage or one engine at a time and can only be entered from the right. If train A has 3 carriages and train B has 4 carriages, how can they pass with the minimum number of moves?

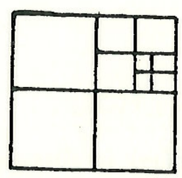
Solution by M. Garth (Castle Hill High), slightly condensed.

Assuming that a carriage cannot be pushed by hand, and that carriages can be hitched to the front of an engine, this is the minimal solution. The carriages are labelled A1, A2, A3 and B1, B2, B3, B4. A move consists of any part or all of the train(s) moving in one direction.

Moves: (1) Engine A moves past then (2) into the siding. (3) Train B moves left past the siding. (4) Engine A leaves the siding. (5) Engine B moves right past the siding, pulling A1, A2 and A3, (6) pushes A3 into the siding, then (7) reverses leaving A3 there. (8) train B moves left past the siding pushing A1 and A2. (9) Engine A connects to A3 and (10) pulls A3 out of the siding. Moves (11)-(16): repeat moves (5)-(10) with carriage A2. Moves (17)-(22): repeat moves (5)-(10) with carriage A1.

Also solved by B. Little (Castle Hill High) and P. Stott (Newington).

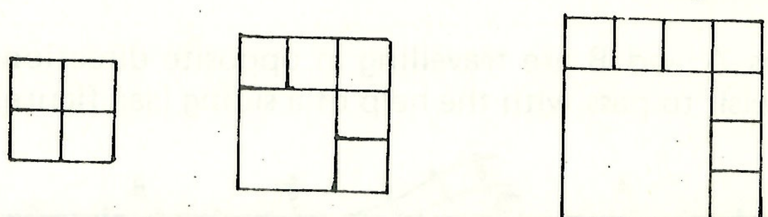
334. It is impossible to dissect a square into 2 smaller squares, but it can be dissected into 10 smaller squares as shown in the figure.



Find all numbers n such that it is impossible to dissect a square into n squares.

Solution:

If a dissection into n squares is possible, then by quadrisecting one of the n squares one obtains a dissection into (n+3) squares. The diagrams



show dissections into 4, 6 and 8 squares. The only integers greater than 1 which cannot be obtained by adding a multiple of 3 to one of these are 2, 3 and 5. To save space, I do not supply here the easy proofs that no dissection is possible for these values of n.

Editor's comment:

Strangely, although this does not seem nearly as difficult as some of the other problems for which correct answers were received, no perfect solutions reached me. D. Dowe (Geelong Grammar, P. Stott, M. Garth, and S. Dunlop (Abbotsleigh) found most dissections, but all managed to overlook some.

335. A school held a special examination to decide which student in year 12 was best overall in the subjects English, History, French, Maths and Science. Five students – Alan, Barbara, Charles, David and Evonne – sat for five papers, one in each of these five subjects. To simplify matters, the top student in a paper was given 5 marks, the next student was given 4 marks, and so on, the last student in a paper being awarded 1 mark (fortunately, no two students tied in any of the papers). When the marks for each student were collected, the following facts were noted:

(1) Alan had an aggregate mark of 24;

(2) Charles had obtained the same mark in four out of the five subjects.

(3) Evonne, the Mathematician, had topped Mathematics, although she only came third in Science.

(4) The students' aggregate marks were in alphabetical order, and no two students had the same aggregate.

What we want to know is:

(a) What was Barbara's mark in Maths?

(b) How many of the 5 students obtained the same mark in at least four out of the five subjects? (Charles was one of these!)

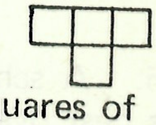
Solution by P. Stott (Newington College):

Since Alan had an aggregate of 24, he must have had four firsts and a second; and since Evonne topped in Maths, it was in this subject that he must have come second. Note also that Evonne, the lowest scorer, got at least 11. This leaves at most 40 of the original 75 marks awarded to be distributed between Barbara, Charles and David. Since there were no ties, the only possible arrangement of aggregates is: Charles 24, Barbara 15, Charles 13, David 12 and Evonne 11.

Since Charles got the same mark in four subjects, and since he cannot have scored a 5, he must have four 3's and a 1 in his 13 points; since Evonne scored 3 in Science, he must have got his 1 there. This means that all the odd scores except the 1 for Maths have been accounted for; and since Barbara had an odd aggregate, she must have got 1 for Maths.

The only possible solutions therefore are: Alan, four 5's and a 4, Barbara, three 4's, a 2 and a 1, Charles, four 3's and a 1, David four 2's and a 4, and Evonne three 1's, a 5 and a 3. Thus, Barbara got 1 in Maths, and three students received the same mark in at least four of the subjects.

Also solved by P. Crump (Sydney Grammar), D. Dowe, M. Garth, J. Taylor (Woy-Woy High), and S. Dunlop.

336. You are given an 8 x 8 chessboard and 16 tiles of the shape  where each of the squares in the T-shape is the same size as the squares of the Chessboard.

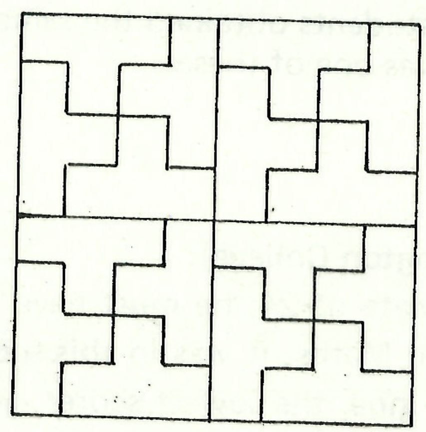
(i) Can the Chessboard be completely covered with these tiles?

(ii) If one of the T-shaped tiles were replaced by a square tile which just covers four of the chessboard squares, can the chessboard be completely covered by these 16 tiles?

In each case, you must either show how to cover the board, or prove that it is impossible.

Solution by P. Stott (Newington College):

(i) Yes, the board can be covered using the pattern shown below. Note it is the same pattern repeated in each of the four quarters of the board.



(ii) No. Note that a T-shaped piece must cover either three black and one white or three white and one black squares. Since a square piece must cover two white and two black squares, it is impossible to cover a chessboard with these 16 pieces, as a chessboard has 32 white and 32 black squares. (i.e. let the number of T-tiles with 3-black 1-white be x;

$3x + (15-x) + 2 = 22$; $2x = 32-2-15$; $x = 7\frac{1}{2}$ and is non-integral).

Also solved by P. Crump, D. Dowe, M. Garth, B. Little, and S. Dunlop. The following also sent correct answers, but I considered their discussion of the second part somewhat less convincing. K. Lam (Trinity Grammar), Carolyn Sue (St Catherine's), J. Taylor.

337. x and y are real numbers such that $x + y = 1$ and $x^4 + y^4 = 7$. Find $x^2 + y^2$ and $x^3 + y^3$.

Solution by P. Crump (Sydney Grammar), condensed to save space:

Set $x^2 + y^2 = A,\ xy = B$.

Squaring $x + y = 1$, we obtain

(i) $A + 2B = 1$

From $x^4 + y^4 = 7$, we easily obtain

(ii) $A^2 - 2B^2 = 7$.

Eliminating B from (i) and (ii) yields

$$A^2 + 2A - 15 = 0$$

whence $A = 3$ or -5. The negative value must be discarded since A is the sum of squares of real numbers. So $A = 3$, and from (i), $B = -1$. Then

$$\begin{aligned} x^3 + y^3 &= (x + y)^3 - 3xy(x + y) \\ &= 1 - 3B \\ &= 4 \end{aligned}$$

Thus $x^2 + y^2 = 3,\ x^3 + y^3 = 4$.

Also solved by P. Stott, and D. Dowe.

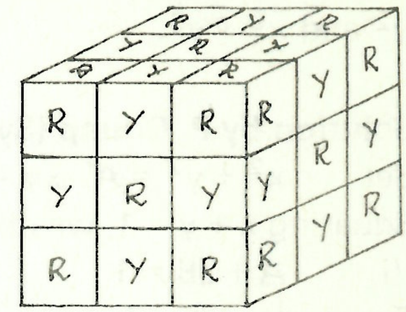
338. You are given 216 blocks each of dimensions 1 cm x 1 cm x 8 cm. Is it possible to build a cube of dimensions 12 cm x 12 cm x 12 cm with these blocks? (As in problem 336, you must either show how to do it, or prove that it is impossible.)

P. Stott (Newington College) writes:

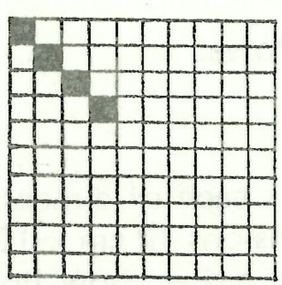
No, it is impossible to do this. Note that $(8 + 4)^3 = 8^3 + 3(8^2.4) + 3(8.4^2) + 4^3$, where the part in brackets is a rectangular prism or cube with the dimensions indicated. Since all parts have a dimension of eight at least once except the 4^3, it can be seen that all rectangular prisms can be constructed using the 8x1x1 blocks; however, no matter how the prisms and cubes are arranged in the 12x12x12 cube, there will be a cube measuring 4x4x4 which cannot be filled with the blocks inside the 12x12x12 cube.

This is plausible, but is it quite satisfying as a proof? I find myself left uneasy that there may be some arrangement of the blocks which fills out the 12x12x12 cube, but does not fill out neatly any 8x8x8 smaller cube within it, for example.

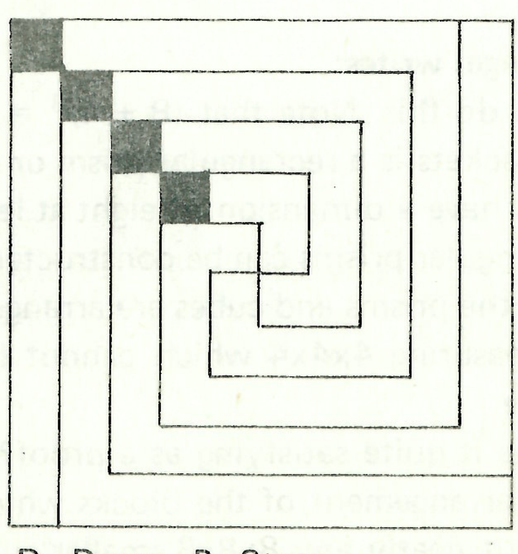
Imagine the 12x12x12 cube composed of 27 cubes of side 4 cm, each of either red wood or yellow wood in a 3-dimensional "checkerboard" arrangement. (There are 14 red ones, and 13 yellow). If each of the 4-cm cubes is further cut up into 64 1-cm cubes every horizontal or vertical row of the 12-cm cube consists of 4R, 4Y, 4R or else 4Y, 4R, 4Y small cubes. In whatever position in any row or column an 8x1x1 block is placed, it will be compsosed of exactly 4 red cubes, and 4 yellow ones. [e.g. It could be made up of 1 yellow cube, then 4 red ones, then the other 3 yellow ones] It is clear that the maximum number of such blocks which can be present is ¼ x (total number of small yellow cubes) = ¼ x 13 x 64; and that then 64 small red cubes will be left over, justifying P. Stott's assertion.



339. Show how to dissect the square in the figure into 4 congruent pieces, each containing one of the black squares.

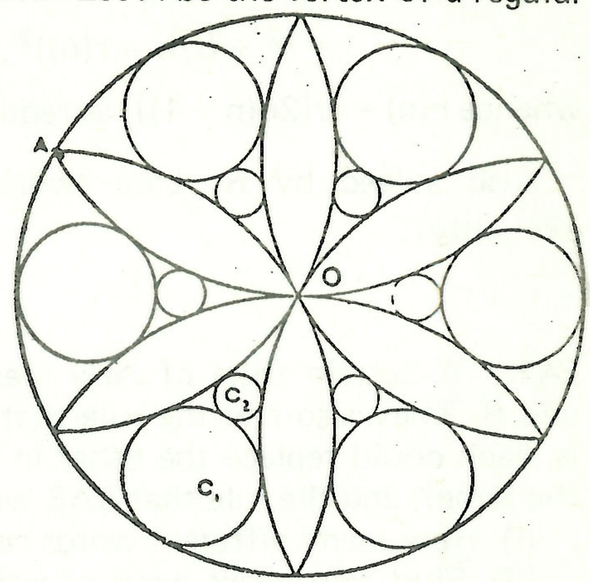


Solution by M. Garth (Castle Hill High).



Also solved by K. Lam, D. Dowe, P. Stott.

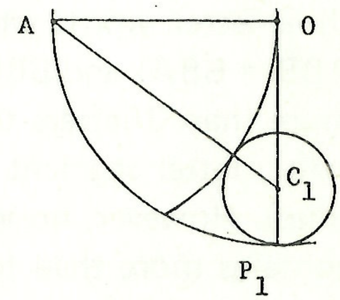
340. Let O be the centre of a circle C of radius r. Let A be the vertex of a regular hexagon inscribed in C. Using A and the other vertices of the hexagon as centres, arcs of radius r are drawn as in the figure. The result is the six-petaled "flower" of the figure. Next are drawn the largest circles which will fit between petals, for example C_1. Then is drawn the next largest C_2, and so on (remaining not drawn). What are the radii of the circles C_1, C_2, C_3, and so on?



The following solution uses the same method as that of A. Fekete (Sydney Grammar):

The radius r(n) of the n'th circle is r/[2n(n+1)], which we now prove by induction:

n = 1. In the right-angled triangle AOC_1,



$$AC_1^2 = AO^2 + OC_1^2,$$

or,

$(r + r(1))^2 = r^2 + (r - r(1))^2$,

whence $r(1) = r/4 = r/(2 \times 1 \times (1 + 1))$.

Assume that $r(k) = r/(2k(k + 1))$ for $k = 1,2,\ldots, n-1$.

Then $P_1P_n = P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n$

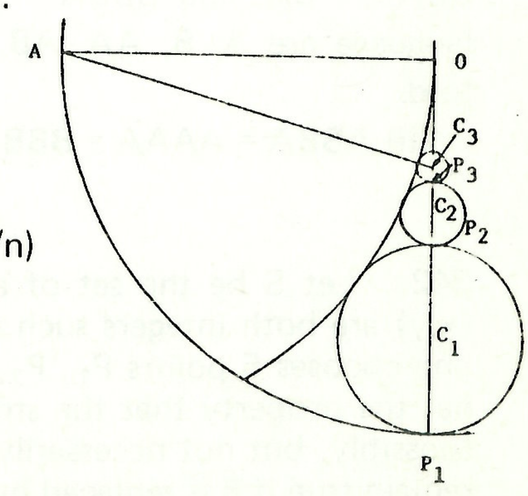
$$= 2r(1) + 2r(2) + \ldots + 2r(n-1)$$

$= r/(1 \times 2) + r/(2 \times 3) + \ldots + r/((n-1)n)$

$= r(1/1 - 1/2) + r(1/2 - 1/3) + \ldots + r(1/(n-1) - 1/n)$

$= r(1 - 1/n)$.

So $OP_n = r - P_1P_n = r/n$.



In the right-angled triangle AOC_n,

$AC_n^2 = AO^2 + OC_n^2$,

$$\text{or } (r + r(n))^2 = r^2 + (OP_n - r(n))^2$$
$$= r^2 + (r/n - r(n))^2,$$

whence $r(n) = r/(2n(n+1))$, as required.

Also solved by R. Lam (working not supplied), P. Stott ($r(n)$ not found explicitly).

341. A certain tribe of early men had an alphabet consisting of two letters A and B. They also had the rule that in any word ABA was equivalent with B (that is, each could replace the other in the word and the word was considered to be the same); and the rule that BAB was equivalent to A.

(i) How many different words could be represented?

(ii) Find two other ways of writing down a certain 4-lettered current "pop" group whose name may be written in this language.

Solution by P. Stott (Newington College):

(i) There are two one-letter possibilities (A and B) and three different two-letter possibilities. Note that AA = ABAB = BB. Thus, there are only two three-letter words which are different and which cannot be simplified: AAA (= ABB = BBA) and BBB (= BAA = AAB). (ABA = B and BAB = A). So any word of more than 3 letters that cannot be simplified contains either all A's or B's, or has a four-letter segment of AAAB or BBBA (possibly after changing according to the rules). However, no word can be different from all words with fewer letters if it contains more than four A's or B's in it together, for AAAAA = BBABB = BAB = A, and BBBBB likewise = B. (i.e. the word can be simplified). Moreover, AAAB = BBAB = BA, and BBBA = AB. Thus, the only different words in this primitive language are A, B, AA, AB, BA, AAA, BBB, and AAAA, 8 different words all told.

(ii) ABBA = AAAA = BBBB.

342. Let S be the set of all points in the Cartesian plane whose coordinates (x,y) are both integers such that $0 \leqslant x \leqslant 100$, $0 \leqslant y \leqslant 100$. Show that however one chooses 5 points P_1, P_2, P_3, P_4, P_5 from S, at least one pair of these points has the property that the straight line through them contains a third point of S (possibly, but not necessarily, another of the chosen points). Does the statement remain true if 5 is replaced by 4?

Solution by D. Dowe (Geelong Grammar), slightly condensed:

Of the five x-coordinates there are at least three with the same parity (i.e. all odd or all even). If we consider the three corresponding y-coordinates, at least two have the same parity. Thus we can find two points A(x(A), y(A)) and B(x(B), y(B)) such that x(A) + x(B) and y(A) + y(B) are both even. Then the midpoint of AB, ((x(A) + x(B))/2, (y(A) + y(B))/2) has integral coordinates, so is in the set S.

If 5 is replaced by 4, the statement is no longer true. For example, (0,1), (1,0), (0,0), (0,1).

Also solved by P. Stott, and A. Fekete.

343. We define a "shuffle" of a deck of N cards numbered 1, 2, . . . , N to be a specific procedure for arranging them in a different order. If one systematically repeats the same shuffle of the deck enough times, it returns to its original order. For instance, if the shuffle consists of interchanging the top 2 cards of the deck it returns to its original order after 2 shuffles. What shuffle of a deck of 28 cards requires the largest number of repetitions before returning to the original order?

Solution by A. Fekete (Sydney Grammar):

Any cycle is a permutation, which can be written as a product of disjoint cycles. The sum of the lengths of these cycles must be 28, and the pack will return to its original order after a number of shuffles equal to the Lowest Common Multiple of the lengths of the cycles.

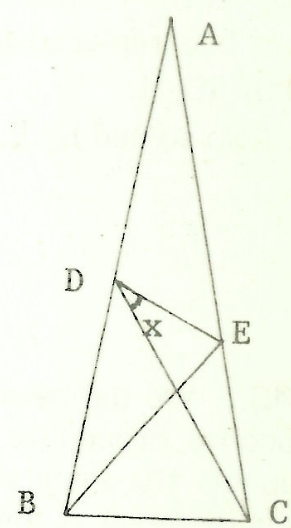
In the maximal-return-time shuffle, there can be no cycle whose length is a composite number, unless this is a power of a prime (e.g. 8,9,25). This is because any other composite number can be written as the product of two numbers with no common factor. The cycles with these lengths will return to the original pack in the same number of shuffles as the initial composite number, but will have a lower sum, leaving some other cycle to increase the L.C.M.

Thus the best shuffle will be a product of cycles whose lengths are in the set {1,2,3,4,5,7,8,9,11,13,16,17,19,23,25}. By trial and error the best combination is 2,3,5,7,11, with LCM 2310.

Also solved by D. Dowe, and P. Stott.

344. *(Suggested by Mr G. Davis of North Sydney Tech. College).* In the following diagram $AB = AC$, $\angle DAE = 20^\circ$, $\angle DCB = 60^\circ$, $\angle EBC = 50^\circ$ and $\angle CDE = x^\circ$. Find x, without using trigonometric tables.

Solution by P. Bos (Sydney Boys High):



In ΔDEC, $\sin\angle DEC/DC = \sin\angle EDC/EC$, or,
$\sin(160^\circ - x)/DC = \sin x/EC$. (i)

In ΔDBC, $\sin\angle CDB/BC = \sin\angle CBD/CD$, or,
$\sin 40^\circ/BC = \sin 80^\circ/CD$. (ii)

From (i) and (ii),
$\sin(160^\circ - x)/\sin x = \sin 80^\circ/\sin 40^\circ$,
since $BC = EC$ (ΔBEC is isosceles).

So $\sin(x + 20^\circ)/\sin x = \sin 80^\circ/\sin 40^\circ$,

$(\sin x \cos 20^\circ + \cos x \sin 20^\circ)/\sin x = 2\cos 40^\circ$,

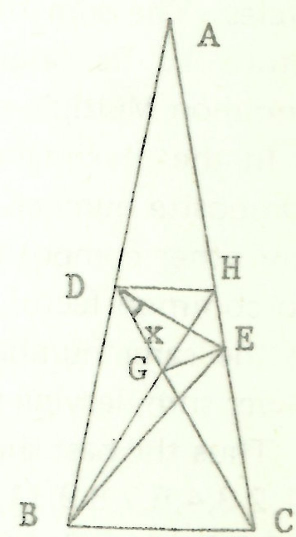
$\sin 20^\circ \cos x = (2\cos 40^\circ - \cos 20^\circ)\sin x$,

$$\begin{aligned}\tan x &= \sin 20^\circ/(2\cos 40^\circ - \cos 20^\circ)\\ &= \sin 20^\circ/(2\cos 40^\circ - (\cos 40^\circ + \cos 80^\circ))\\ &= \sin 20^\circ/(\cos 40^\circ - \cos 80^\circ)\\ &= \sin 20^\circ/(2\sin 20^\circ \sin 60^\circ)\\ &= 1/\sqrt{3},\end{aligned}$$

so $x = 30^\circ$.

Solution by M. Reynolds (Marist Bros, Pagewood). Brief outline:

Construct $\angle HBC = 60^\circ$. Join DH, GE. (See fig.) Prove ΔBCG and ΔDGH are equilateral.



Since $CE = CB = CG$ we have $\angle CGE = \angle GEC = 80^\circ$. We can now show that $\angle EGH$ and $\angle EHG$ are each 40°, whence $EG = EH$. Now ΔEGD and ΔEHD are congruent (three sides theorem) and DE bisects the 60° angle $\angle HDG$. Thus $\angle EDG = 30^\circ$.

Also solved by P. Stott, and A. Fekete.