SEQUENCES GENERATED BY POLYNOMIALS

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Consider the quadratic x^2+x+1. By substituting $x=1,2,3,\ldots$ we can form a sequence

$$x^2 + x + 1 : 3\ 7\ 13\ 21\ 31 \ldots$$

If we were presented with the sequence 3,7,13,21,31,... it would not, however, be obvious what the polynomial was that generated it. Indeed we could not assume that it was generated by a polynomial at all. I have recently discovered a method by which this problem can be resolved.

Let us consider the differences between successive terms in the sequence.

$x^2 + x + 1$:	3	7	13	21	31	...
1st diff. :	4	6	8	10		...
2nd diff. :		2	2	2		...

Note that the 1st differences form an A.P. which is generated by $(2x+2)$, a polynomial of degree 1. Note also, that the 2nd differences are given by (2), a polynomial of degree 0. Why the polynomial $(2x+2)$ gives the 1st differences and (2) gives the 2nd differences is not altogether clear. Let us consider some more polynomials.

$2x^2 + x + 2$:	5	12	23	38	...
$4x + 3$:	7	11	15		...
4 :	4	4			...

$3x^2 + 2x + 1$:	6	17	34	57	...
$6x + 5$:	11	17	23		...
6 :	6	6			...

It is apparent that the coefficient of x^2 affects the coefficient of x in the 1st differences and that the coefficients of x^2 and x both affect the constant in the 1st differences. A reasonable hypothesis for a general quadratic and its differences would be:

Quadratic : $a_2x^2 + a_1x + a_0$
1st diff. : $2a_2x + (a_2 + a_1)$
2nd diff. : $2a_2$

This is easily proved :

Let $P(x) = a_2x^2 + a_1x + a_0$ be the polynomial
and let $P^*(x) = b_1x + b_0$ be its difference.

Now $P(2) - P(1) = P^*(1)$.

$\therefore (4a_2 + 2a_1 + a_0) - (a_2 + a_1 + a_0) = b_1 + b_0$.

$\therefore 3a_2 + a_1 = b_1 + b_0$. (1)

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Now $P(3) - P(2) = P^*(2)$.

$\therefore (9a_2 + 3a_1 + a_0) - (4a_2 + 2a_1 + a_0) = 2b_1 + b_0$.

$\therefore 5a_2 + a_1 = 2b_1 + b_0$. (2)

Solving (1) and (2) simultaneously gives $b_1 = 2a_2$ and $b_0 = a_1$ and the proof is complete.

It is possible to extend this idea of finding the polynomial which generates the 1st differences to polynomials of higher degree and in doing so an interesting relationship is discovered. The coefficients of the differences are dependent on Pascal's Triangle. For a polynomial of degree 3:

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$
$$P^*(x) = 3a_3x^2 + (3a_3 + 2a_2)x + (a_3 + a_2 + a_1)$$

For a polynomial of degree 4:

$$P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$
$$P^*(x) = 4a_4x^3 + (6a_4 + 3a_3)x^2 + (4a_4 + 3a_3 + 2a_2)x + (a_4 + a_3 + a_2 + a_1)$$

For a polynomial of degree 5:

$$P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$
$$P^*(x) = 5a_5x^4 + (10a_5 + 4a_4)x^3 + (10a_5 + 6a_4 + 3a_3)x^2 + (5a_5 + 4a_4 + 3a_3 + 2a_2)x + (a_5 + a_4 + a_3 + a_2 + a_1)$$

And generally for a polynomial of degree n:

$P(x)$	$P^*(x)$
a_0	$\{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots + a_n\}$
$+ a_1x$	$+ x\{2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + \ldots + na_n\}$
$+ a_2x^2$	$+ x^2\{3a_3 + 6a_4 + 10a_5 + 15a_6 + 21a_7 + \ldots\}$
$+ a_3x^3$	$+ x^3\{4a_4 + 10a_5 + 20a_6 + 35a_7 + \ldots\}$
$+ a_4x^4$	$+ x^4\{5a_5 + 15a_6 + 35a_7 + \ldots\}$
$+ a_5x^5$	$+ x^5\{6a_6 + 21a_7 + \ldots\}$
$+ a_6x^6$	$+ x^6\{7a_7 + \ldots\}$
$+$	$+$
.	.
.	.
.	.
$+ a_{n-4}x^{n-4}$	$+ x^{n-4}\{(n-3)a_{n-3} + \frac{(n-2)(n-3)}{2.1}a_{n-2} + \frac{(n-1)(n-2)(n-3)}{3.2.1}a_{n-1} + \frac{n(n-1)(n-2)(n-3)}{4.3.2.1}a_n\}$
$+ a_{n-3}x^{n-3}$	$+ x^{n-3}\{(n-2)a_{n-2} + \frac{(n-1)(n-2)}{2.1}a_{n-1} + \frac{n(n-1)(n-2)}{3.2.1}a_n\}$
$+ a_{n-2}x^{n-2}$	$+ x^{n-2}\{(n-1)a_{n-1} + \frac{n(n-1)}{2.1}a_n\}$
$+ a_{n-1}x^{n-1}$	$+ x^{n-1}\{na_n\}$
$+ a_nx^n$	

The relationship between the coefficients and Pascal's Triangle is now evident. By taking strips at an angle of 45° the coefficient of any power of x in $P^*(x)$ is found

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
x^6 x^5 x^4 x^3 x^2 x^1 x^0

The formula for $P^*(x)$ when $P(x)$ is a polynomial of degree n can now be stated in terms of the coefficients of Pascal's Triangle. I will use the familiar ${}^nC_r$ notation, where ${}^nC_r$ is the rth term of the nth row [the opening term of any row is the zeroth term; the row consisting of 1 1 is considered the 1st row].

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r(r-1)(r-2)\ldots 3.2.1}$$

P(x)	P*(x)
a_0	$\{{}^1C_1a_1 + {}^2C_2a_2 + {}^3C_3a_3 + {}^4C_4a_4 + \ldots + {}^nC_na_n\}$
$+\ a_1x$	$+\ x\{{}^2C_1a_2 + {}^3C_2a_3 + {}^4C_3a_4 + \ldots + {}^nC_{n-1}a_n\}$
$+\ a_2x^2$	$+\ x^2\{{}^3C_1a_3 + {}^4C_2a_4 + \ldots + {}^nC_{n-2}a_n\}$
$+\ a_3x^3$	$+\ x^3\{{}^4C_1a_4 + \ldots + {}^nC_{n-3}a_n\}$
$+\ a_4x^4$	
+	+
.	.
.	.
.	.
$+\ a_{n-3}x^{n-3}$	$+\ x^{n-3}\{{}^{n-2}C_1a_{n-2} + {}^{n-1}C_2a_{n-1} + {}^nC_3a_n\}$
$+\ a_{n-2}x^{n-2}$	$+\ x^{n-2}\{{}^{n-1}C_1a_{n-1} + {}^nC_2a_n\}$
$+\ a_{n-1}x^{n-1}$	$+\ x^{n-1}\{{}^nC_1a_n\}$
$+\ a_nx^n$	

With this information we may now return to our original problem of finding the polynomial which generates a particular sequence. Let us consider the sequence 4, 31, 130, 373, 856, 1699 . . . and attempt to find the polynomial which generates it. We will use our knowledge of differences to aid us.

We begin by taking successive differences of the polynomial sequence until a sequence of degree 0 is reached. This will tell us what the degree of the polynomial is. Note that the condition for a sequence being generated by a polynomial of degree n is that its nth differences are constant.

$P(x)$:	4		31		130		373		856		1699		...
$P^*(x)$:		27		99		243		483		843		...	
$P^{**}(x)$:			72		144		240		360		...		
$P^{***}(x)$:				72		96		120		...			
$P^{****}(x)$:					24		24		...				

Hence the polynomial that generates the sequence is of degree 4. By using our knowledge of differences, in the reverse sense, we can discover the polynomial required.

$P^{****}(x) = 24$

$\therefore P^{***}(x) = 24x + C_1$.

When $x = 1$, $P^{***}(x) = 72$, so $C_1 = 48$.

$\therefore P^{***}(x) = 24x + 48$.

Remembering that $a_2x^2 + a_1x + a_0$ has 1st diff. $2a_2x + (a_2 + a_1)$, we see that

$$P^{**}(x) = 12x^2 + 36x + C_2$$
$$= 12x^2 + 36x + 24, \text{ since } P^{**}(1) = 72.$$

Remembering that $a_3x^3 + a_2x^2 + a_1x + a_0$ has 1st diff. $3a_3x^2 + (3a_3 + 2a_2)x + (a_3 + a_2 + a_1)$, we see that

$$P^*(x) = 4x^3 + 12x^2 + 8x + C_3$$
$$= 4x^3 + 12x^2 + 8x + 3, \text{ since } P^*(x) = 27.$$

Remembering that $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ has 1st diff. $4a_4x^3 + (6a_4 + 3a_3)x^2 + (4a_4 + 3a_3 + 2a_2)x + (a_4 + a_3 + a_2 + a_1)$,

$$P(x) = x^4 + 2x^3 - x^2 + x + C_4$$
$$= x^4 + 2x^3 - x^2 + x + 1, \text{ since } P(1) = 4.$$

So $x^4 + 2x^3 - x^2 + x + 1$ generates the sequence 4, 31, 130, 373, 856, 1699, ...

This method may seem rather tedious, but in practice most of the working may be omitted and the process is very simple. It may be summarised thus:

(1) Take successive differences until a constant sequence is obtained.

(2) By knowledge of differences, the polynomials generating successive differences may be obtained.

(3) Evaluate the constant by considering the 1st term.

While the proof of the formula for the 1st differences is relatively straight forward for a polynomial of low degree (as was illustrated earlier for a quadratic) a proof for a polynomial of degree n has eluded me. I would be interested to hear of any proofs which might be suggested by others.