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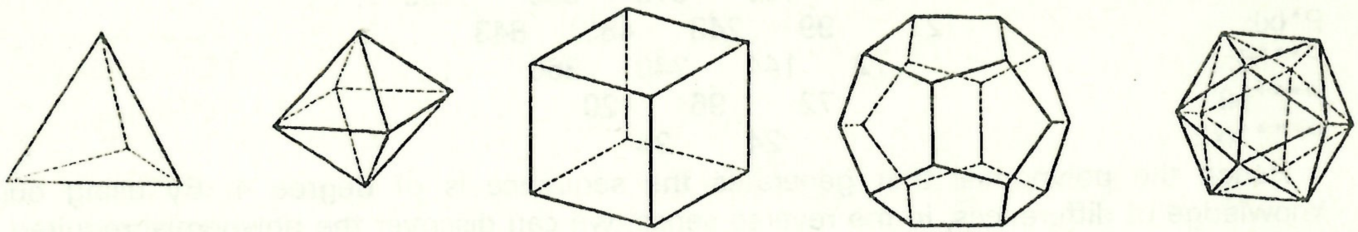

POLYHEDRA
I: THE PLATONIC SOLIDS

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Regular patterns and figures have always played an important part in the civilizations of mankind. The regular polygons and solids (polyhedra) were objects of intense study by the ancient Greeks (circa 300 B.C.) not only because of their geometrical properties but also because they acquired mysterious, semi-religious powers and were held to contain the secrets of the universe. This idea died hard. Centuries later the great Johannes Kepler (1571-1630), whose epoch-making scientific work started modern physics, was still convinced of the truth of these connections. He boldly attempted to relate the regular polyhedra to the real physical world and conceived a remarkable construction to represent the positions and movements of the planets (see figure 1).

Today we know that his model does not represent the solar system, if only because there are more planets than regular polyhedra, so there is no possible simple correspondence of the kind he envisaged.

However, interest and research in polyhedra has continued and is still going on.

The ancient Greeks knew, and could prove, that there exist exactly five regular polyhedra, frequently called the Platonic solids. (A regular polyhedron is a solid whose faces are all the same regular polygon, and with the same number of edges meeting at each vertex.) An interesting and important connection between the number of vertices (corners) V, the number of edges, E, and the number of faces, F, escaped their notice. It was discovered by Leonhard Euler (1707-1783), after whom it is named. It is the formula

$$V - E + F = 2,$$

which we will prove.

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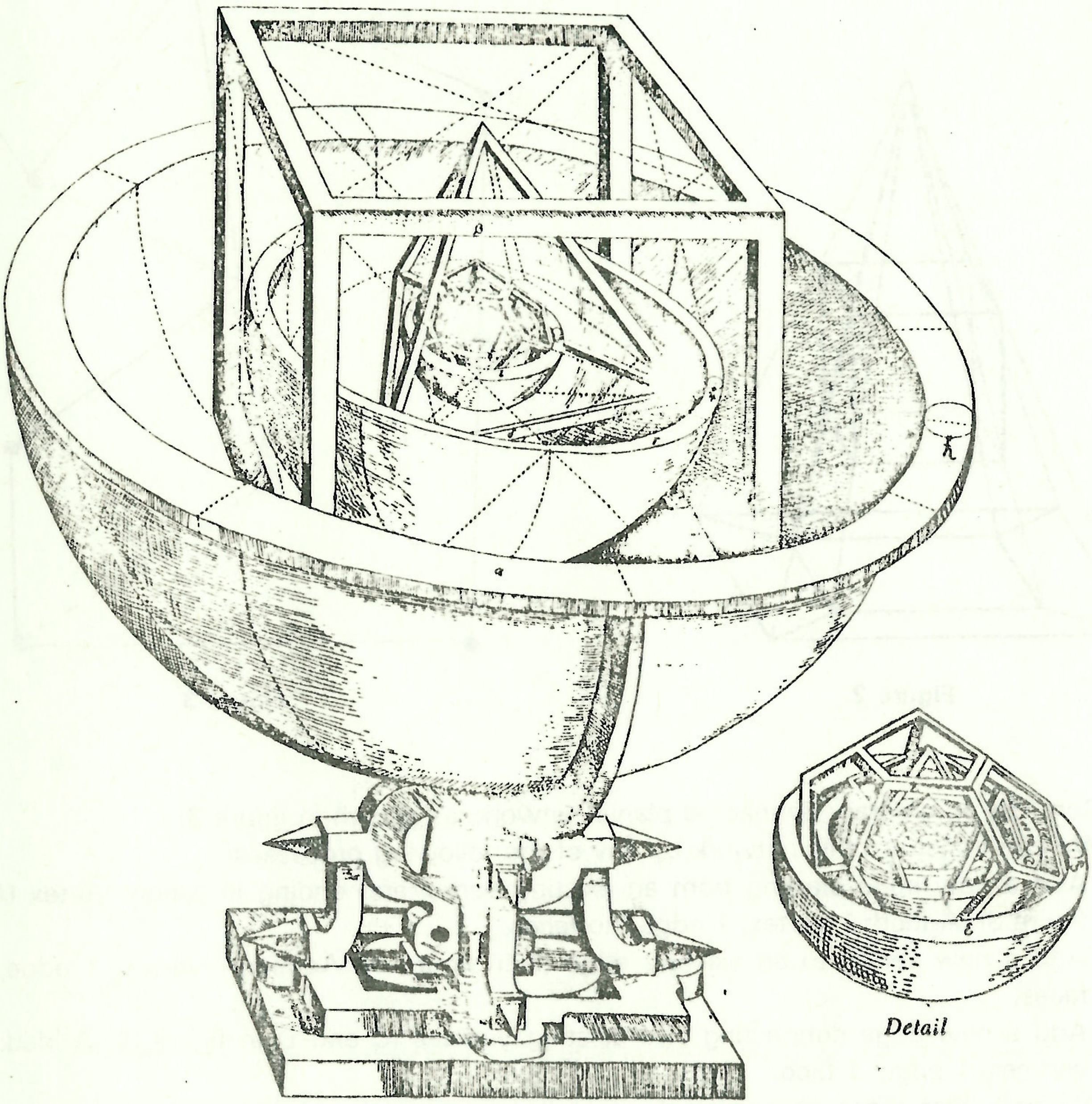

Detail

Figure 1

Consider the skeletal model of a polyhedron and imagine it projected onto a screen in such a way that the light source is reasonably close to, and perpendicularly above, the centre of one of the faces. In figure 2 we show this projection for the cube.

The result is a network in the plane of projection. Each line segment in this network represents an edge and each region a face, with the exception of the face closest to the light source. We compensate for this by counting the surrounding infinite region as a "face". Now we will show that in fact $V-E+F=2$ holds for any connected network in the plane, and so our result is much more general than that which we set out to prove.

A network is a figure in the plane consisting of a finite number of arcs or edges, which meet only at their endpoints, or vertices. The regions into which the network subdivides the plane are called faces.

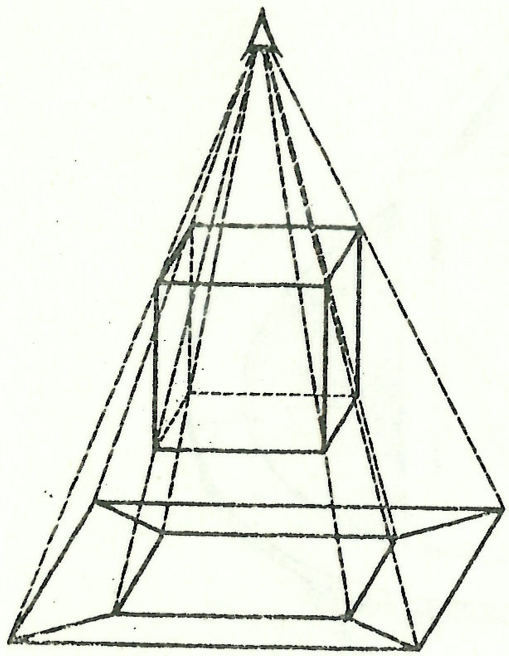

Figure 2

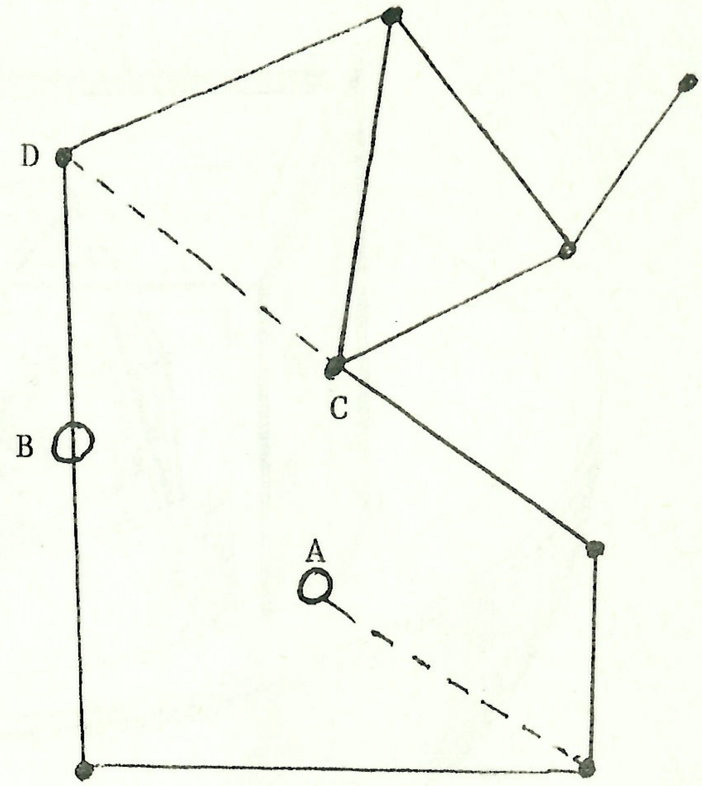

Figure 3

Consider an arbitrary connected planar network, as shown in figure 3.

We can transform this network by any of the following processes:

i) Add a new edge starting from an existing vertex and ending in a new vertex (A in figure 3). Added: 1 vertex, 1 edge, no faces.

ii) Add a new vertex to an existing edge (B in figure 3). Added: 1 vertex, 1 edge, no faces.

iii) Add a new edge connecting two existing vertices (C and D in figure 3). Added: no vertices, 1 edge, 1 face.

We note that these processes do not alter the quantity $V-E+F$. Furthermore, any connected network arises from a single point in the plane by successive applications of the above processes, and for the single point in the plane $V = 1$, $E = 0$, $F = 1$, so $V-E+F = 2$.

With the aid of Euler's Formula, we can show that there are five Platonic solids.

Suppose n edges meet at every vertex. Then, counting every edge twice, we obtain

$$nV = 2E, \text{ so } V = 2E/n.$$

Suppose every face of the polyhedron has r edges. Then, again counting every edge twice, we obtain

$$rF = 2E, \text{ so } F = 2E/r.$$

Substituting these into Euler's Formula, we obtain

$$2E/n - E + 2E/r = 2$$

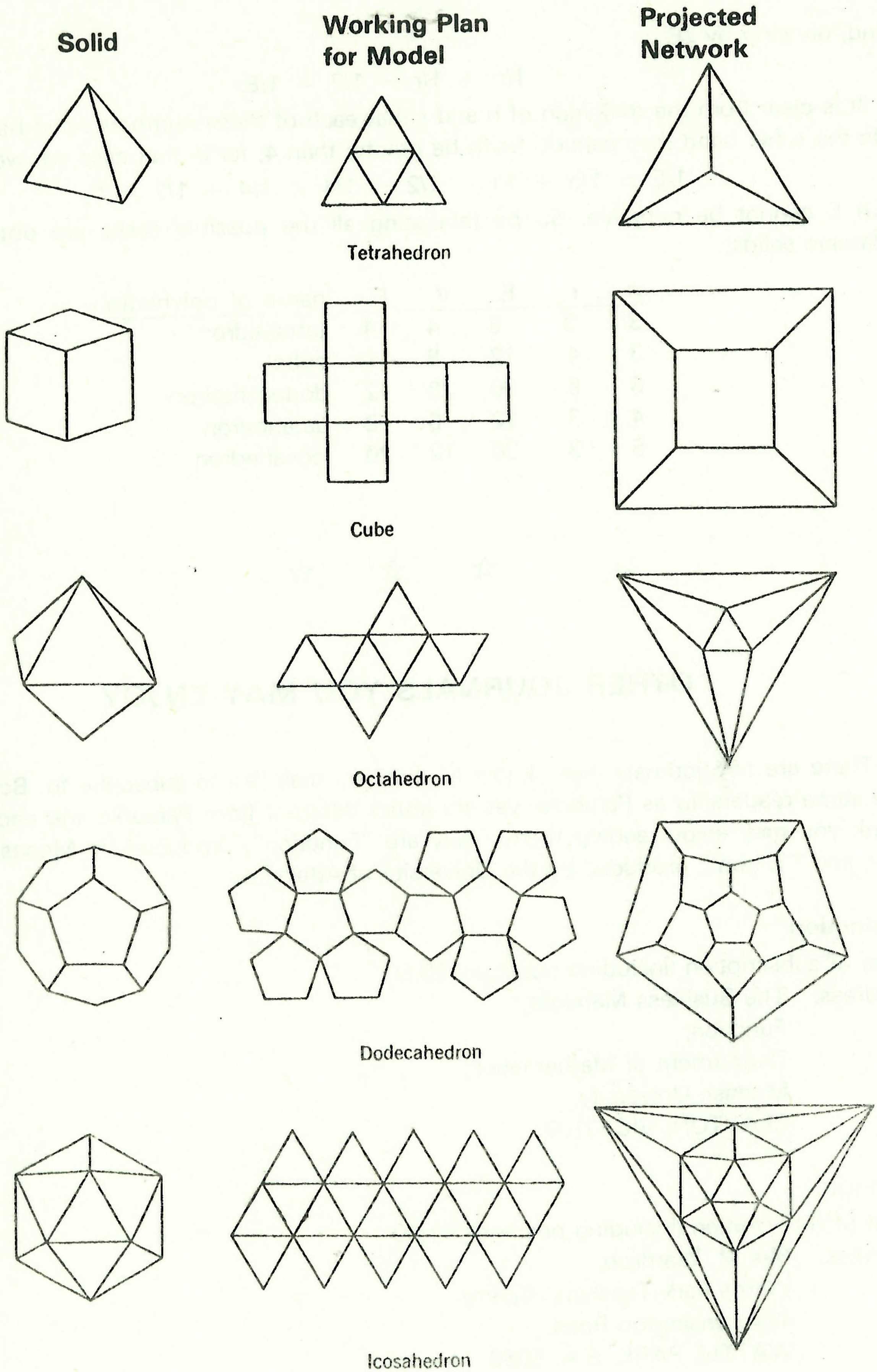
Solid
Working Plan for Model
Projected Network
Tetrahedron
Cube
Octahedron
Dodecahedron
Icosahedron

Figure 4

and, dividing by 2E,

$$1/n + 1/r - 1/2 = 1/E.$$

It is clear from the definition of n and r that each of these numbers must be at least 3. On the other hand they cannot **both** be greater than 4, for in that case we would have

$$1/E = 1/n + 1/r - 1/2 \leq 1/4 + 1/4 - 1/2 = 0,$$

but E cannot be negative. So by tabulating all the possible cases we obtain all the Platonic solids:

n	r	E	V	F	name of polyhedron
3	3	6	4	4	tetrahedron
3	4	12	8	6	cube
3	5	30	20	12	dodecahedron
4	3	12	6	8	octahedron
5	3	30	12	20	icosahedron

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