THE SUMMER SCIENCE SCHOOL

Once again, in early December 1977 the University of N.S.W. held its Summer Science School. Fifty schools throughout the State were invited to send their two best year 11 Science-Maths students to the University for a week's research.

As in 1976, there were two Mathematics projects. One was called "Finding out about Geometry", and was supervised by Professor V.T. Buchwald, the other was "Ordering Pancakes", and was supervised by me.

The six students involved in the first of these were grouped into three pairs. One pair, Jennifer Wheatley of St George Girls High and Mark Gray of Tamworth High worked with Professor Buchwald, and their report of what they learned follows.

M. Hirschhorn

THE EULER'S LINE OF A TRIANGLE

An interesting discovery, and an important theorem in geometry, is that the circumcentre, centroid and orthocentre of a triangle lie on a straight line. This is called the Euler's Line of the triangle.

The right (perpendicular) bisectors of the sides of a triangle are concurrent and their point of intersection is equidistant from the vertices of the triangle. Thus the point of intersection is the centre of a circle which passes through the vertices. The circle is called the circumcircle, and its centre the circumcentre.

To prove that these right bisectors are concurrent, we use any triangle ABC (see figure 1).

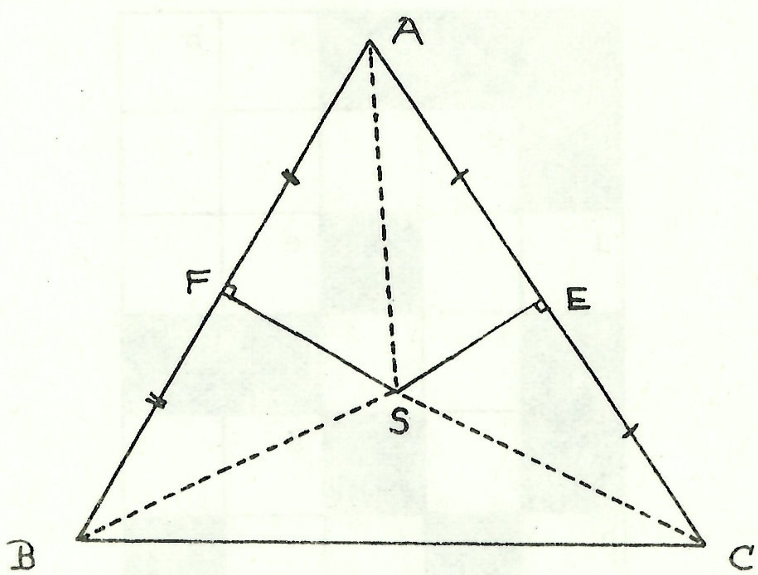

Figure 1

Since A, B, C are not collinear the right bisectors of AB and AC must intersect at a point S.
Since S is on the right bisector of AB,

$$SA = SB$$

(this is because F is the midpoint of AB and the triangles AFS, BFS are therefore congruent).

Since S is on the right bisector of AC,

$$SA = SC$$

(for similar reasons)

$$\therefore SB = SC$$

∴ S is on the right bisector of BC
∴ the right bisectors of AB, BC, CA are concurrent.

The second point mentioned in connection with Euler's Line is the orthocentre. This is where the three altitudes of a triangle concur. The proof that they do so is as follows.

Let ABC be the triangle and AD, BE, CF its altitudes (see figure 2).

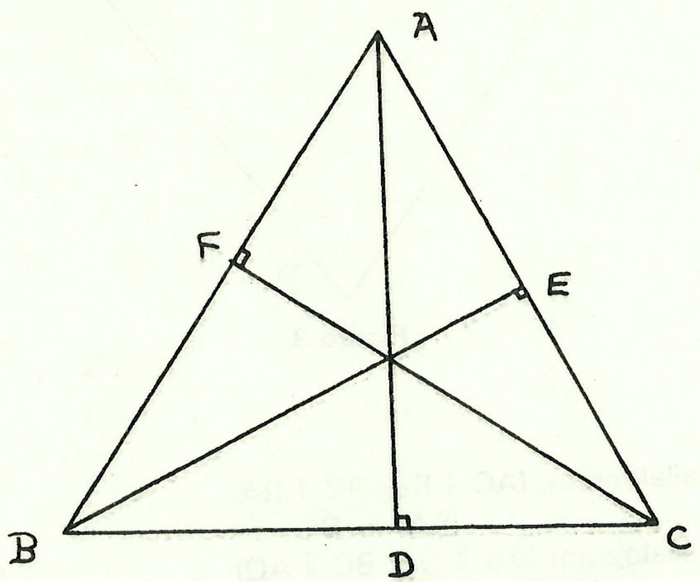

Figure 2

Through A, B, C lines are drawn parallel to BC, AC and AB to form the triangle PQR (see figure 3).

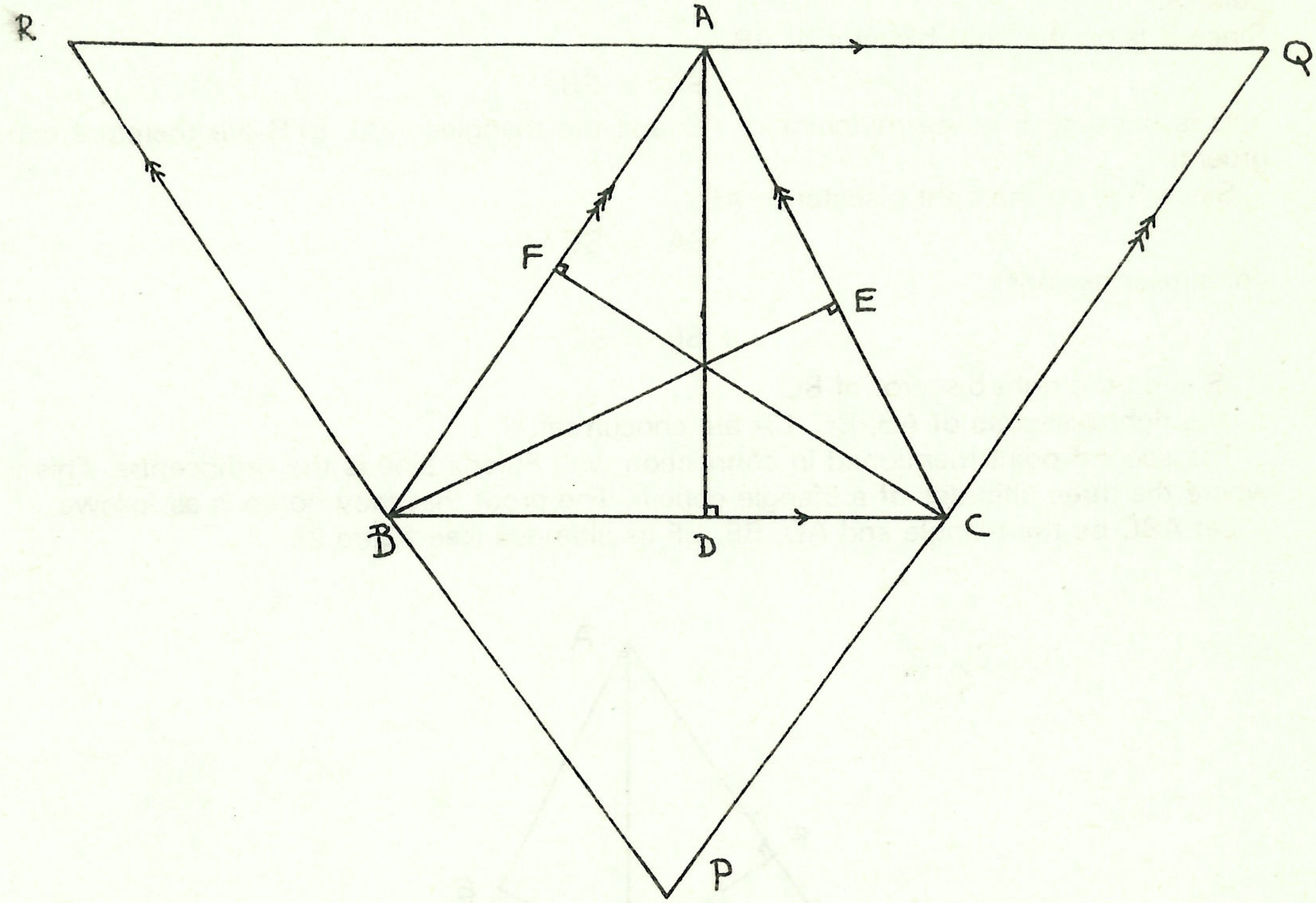

Figure 3

Since RBCA is a parallelogram, (AC ∥ RB; BC ∥ RA)

$$RA = BC$$

Since ABCQ is a parallelogram (AB ∥ QC; BC ∥ AQ)

$$AQ = BC$$

$$\therefore RA = AQ$$

Since RQ is parallel to BC, AD is perpendicular to RQ

$\therefore$ AD is the right bisector of RQ.

Similarly, BE, CF are the right bisectors of RP, PQ. And since the right bisectors of a triangle are concurrent (as proved above), AD, BE and CF are concurrent.

The point at which the altitudes concur is called the orthocentre of the triangle: and the triangle DEF is called the pedal triangle of the triangle ABC.

The third point associated with Euler's line is the centroid. This is the point at which the medians of a triangle concur. The point of intersection is one third of the way along each median from the base to the opposite vertex.

Let the medians AA', BB' of the triangle ABC intersect at G (see figure 4).

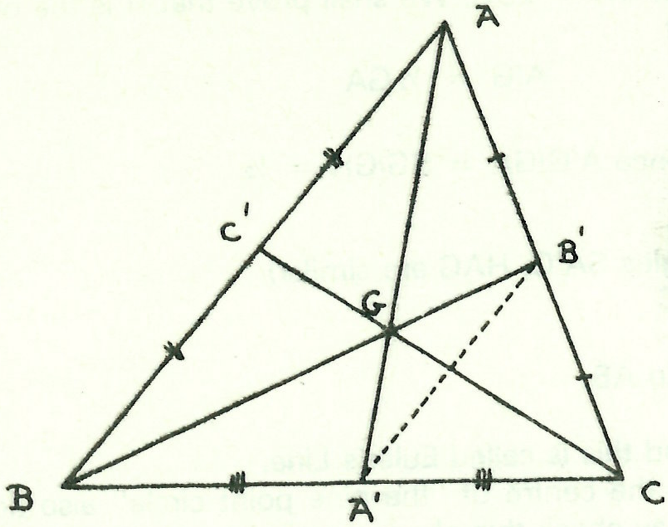

Figure 4

Since CA′ = ½CB and CB′ = ½CA then it is easy to prove triangle CAB, CB′A′ are similar, and hence

A′B′ is parallel to AB and
A′B′ = ½AB.

It follows that the triangles A′GB′, AGB are similar

∴ A′B′/GA = A′B′/AB = ½ or

A′G = ½GA
∴ A′G = ⅓A′A

∴ BB′ cuts AA′ at the point of trisection of AA′ nearest to A′.

Similarly the median CC′ cuts AA′ at this same point. The point is called the centroid of the triangle, and the medians are concurrent at the centroid.

Now that we have seen what each point is, let us look at and obtain Euler's Line.

Taking a triangle ABC, let S and G be the circumcentre and centroid of the triangle. Also A′ and C′ are the mid-points of BC and AB (see figure 5).

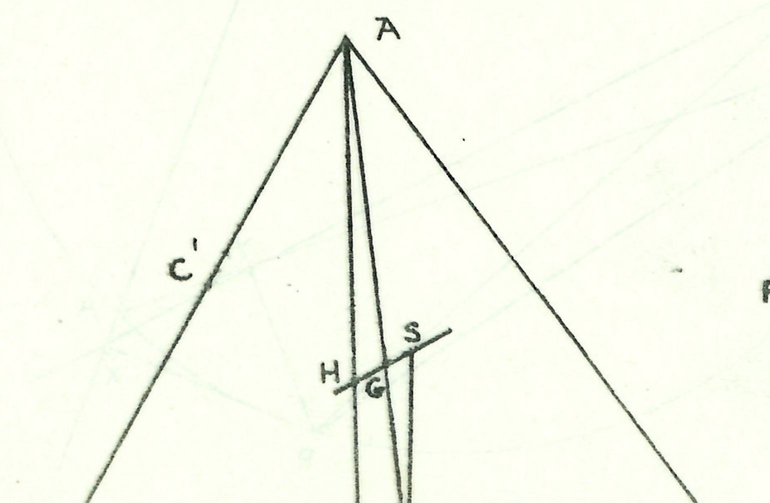

Figure 5

SG is produced to H so that GH = 2SG. We shall prove that H is the orthocentre.
We have shown above that

$$A'G = \tfrac{1}{2}GA$$

$$\text{Hence } A'G/GA = SG/GH = \tfrac{1}{2}$$

∴ AH is parallel to SA′ (triangles SA′G, HAG are similar)
But SA′ is perpendicular to BC
∴ AH is perpendicular to BC
Similarly CH is perpendicular to AB.
H is the orthocentre.
Thus SGH is a straight line, and this is called Euler's Line.

Another point N, known as the centre of "the nine point circle" also lies on SG produced. An interested reader may check this phenomenon in any classical geometry text.

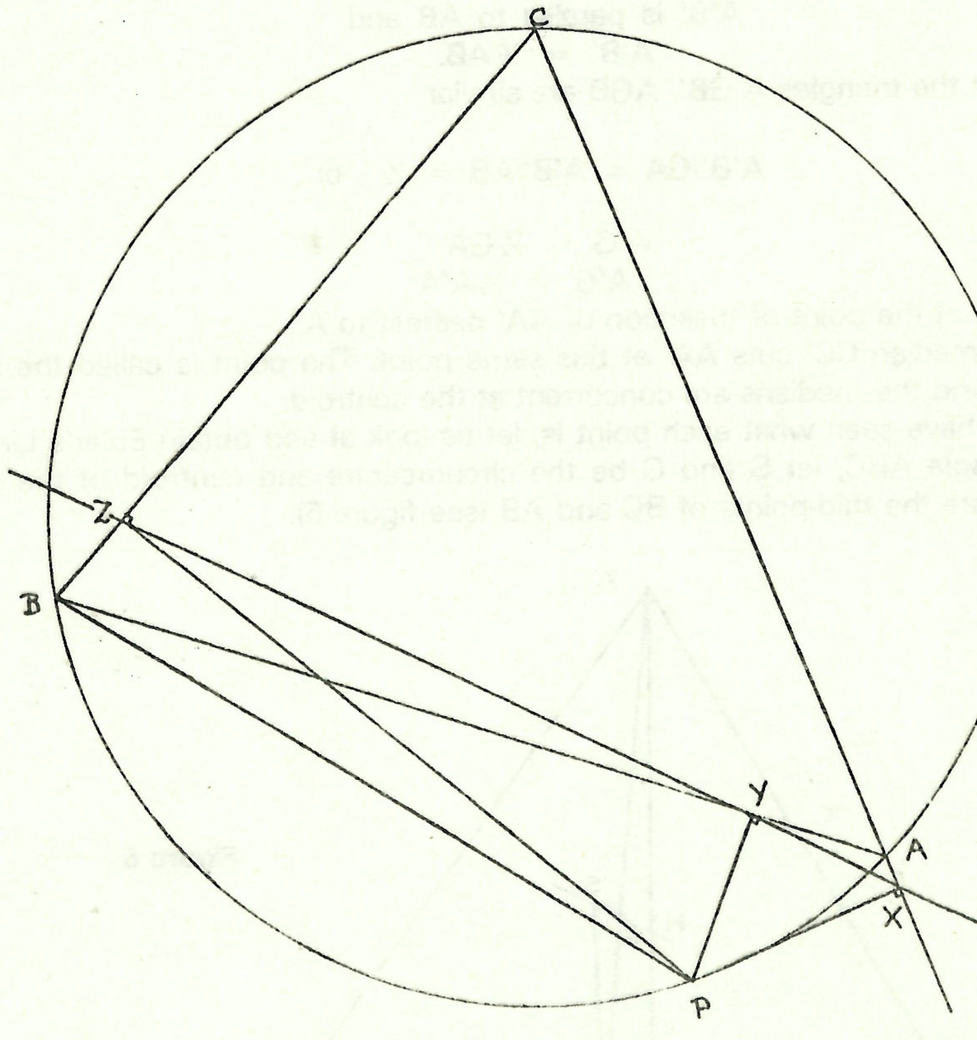

Figure 6