SIMSON'S LINE

While exploring the peculiarities of circles we come upon an interesting phenomenon known as Simson's Line. This line is formed in the following way:

Any triangle — we will call this triangle ABC — is drawn, and a circle drawn such that the vertices A, B and C all lie on the circumference. A fourth point — P — lies anywhere on the circumference of the circle. When perpendiculars are drawn from the point P to the sides AB, AC and BC of the triangle, these perpendiculars meet the sides at X, Y and Z. The phenomenon of Simson's Line is that these 3 points X, Y and Z lie on a straight line (see figure 6).

In proving this Line to actually exist the following basic properties of the geometry of the circle are necessary.

Theorem A. If a quadrilateral is cyclic (i.e. its four vertices lie on a circle) then opposite angles are supplementary. Conversely, if a quadrilateral has opposite angles which are supplementary it is cyclic (see figure 7).

$\angle B + \angle D = 180°$

$\angle A + \angle C = 180°$

(this theorem is proved in any book of basic geometry)

Also, since the opposite angles are supplementary, the exterior angle at B in the diagram is equal to the angle at D.

Theorem B. If a chord is drawn within a circle, this chord subtends equal angles at the circumference of the circle (see figure 8)

$\angle A = \angle B$

DC is a chord of the circle.

Conversely if a quadrilateral is drawn such that two angles subtended by one side are equal, then the quadrilateral is cyclic.

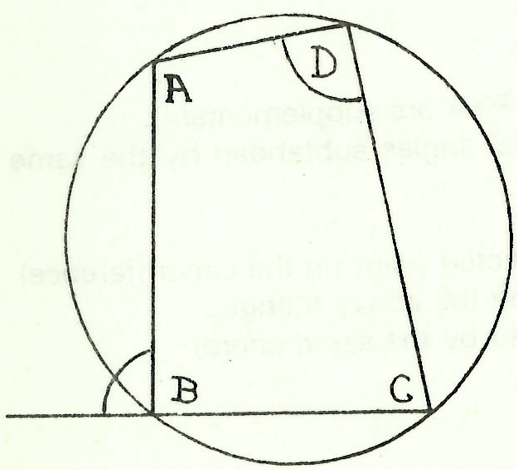

Figure 7

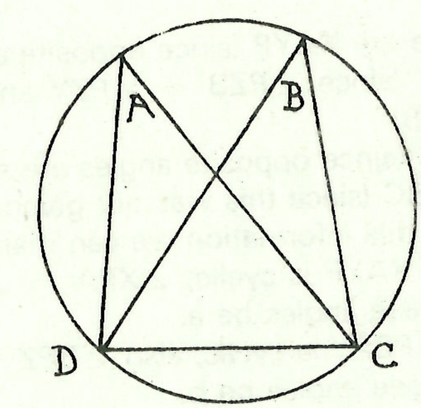

Figure 8

Proof of the existence of Simson's Line:

In the diagram we can discern four cyclic quadrilaterals (see figure 9).

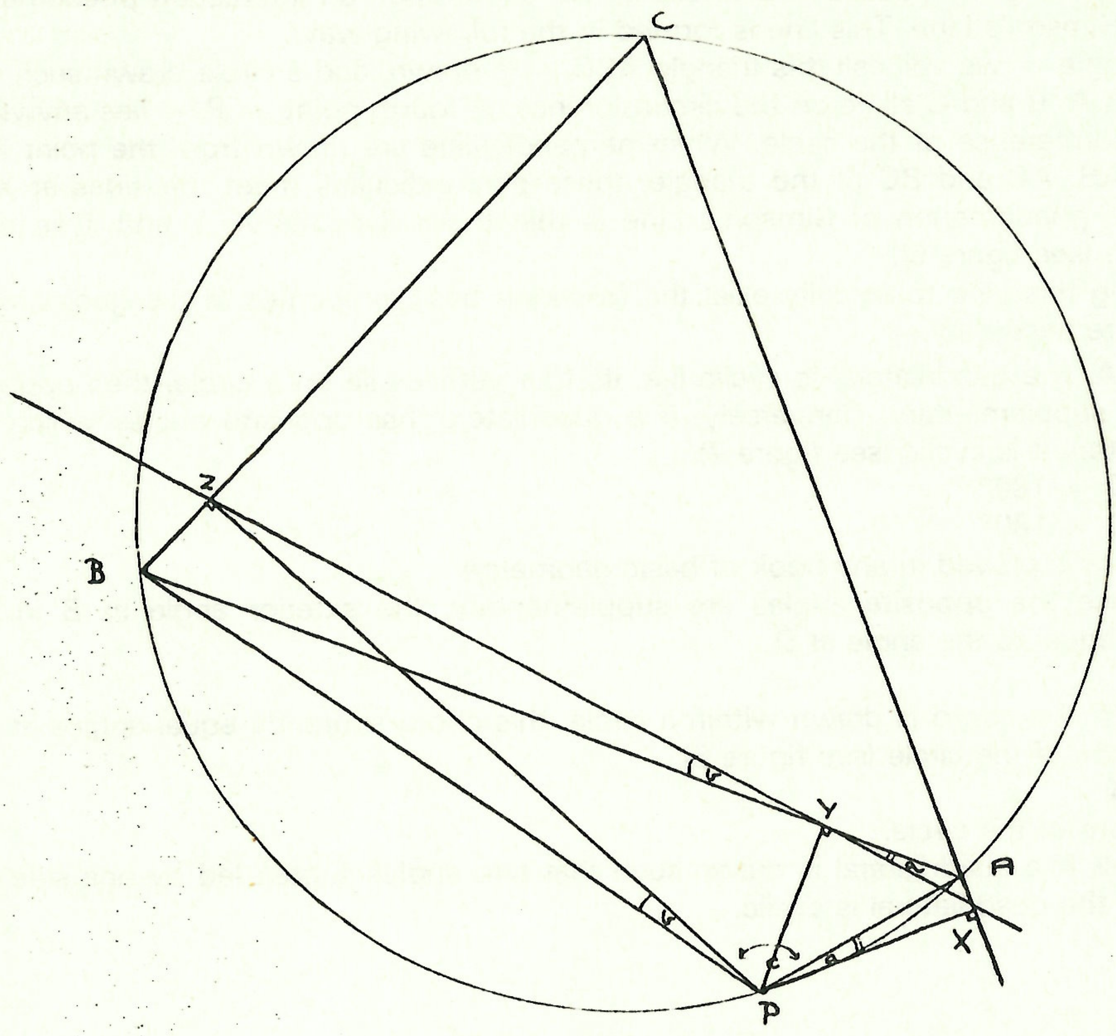

Figure 9

These are XAYP (since opposite angles AXP and PYA are supplementary).

PBZY (since ∠PZB = ∠PZY and these are equal angles subtended by the same interval PB)

XPZC (since opposite angles are supplementary)

and APBC (since this was our given triangle and selected point on the circumference).

From this information we can mark equal angles on the above triangle.

Since XAYP is cyclic, ∠XPA = ∠XYA (subtended by the same chord).

Let these angles be a.

Since PBZY is cyclic, also ∠BPZ = ∠BYZ

Let these angles be b.

Since XPZC is cyclic, ∠XPZ = exterior angle at C.

Also since APBC is cyclic, ∠APB = exterior angle at C.

∴ ∠XPZ = ∠APB.

If we let ∠APZ = c then ∠XPZ = a + c
∠APB = b + c.
∴ a = b.

This also means that the other angles marked a, b i.e. ∠XYA and ∠BYZ are equal.

It is a fact that when two straight lines cross, vertically opposite angles are equal (see figure 10)

Since in our diagram above the angle a = the angle b and AYB is a straight line it follows that XYZ is also a straight line.

Thus the existence of Simson's Line is proved.

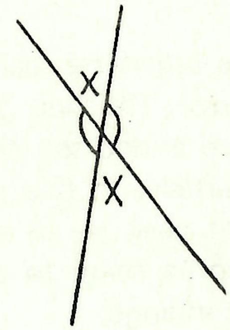

Figure 10

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TRY THIS FOR SIZE

How many digits does 1001^{1000} have?
How many can you find?
Send us your answer.