LETTERS TO THE EDITOR

Simpson's Paradox

Sir,

In the article by Alan Fekete on Simpson's Paradox in Vol. 13 No. 3, there is apparently an error. The first paragraph contains 2 batting scores; 5 for 60 and 3 for 57. When these are added together the total score for Jones' is actually 8 for 117 not 9 for 117 as in the article. In this particular case, therefore the system of adding successive scores does not give rise to an obvious case of Simpson's Paradox, although Smith and Jones' total scores seem to be closer together than one would expect from the scores of individual innings.

The other cases of Simpson's Paradox in the article are correct and indicate the essence of the paradox. Scores that differ in their nature or conditions of test cannot be readily combined in such a simple way without giving rise to some discrepancies.

Ross Baldick,
Year 11,
Chatswood High.

Alan Fekete responds:

Sir,

With regard to the error in my article Vol. 13 No. 3, Jones got in 2nd innings 5 for 70 making him the better bowler with figures of 1:14. Total figures for Jones are thus 10 for 130.

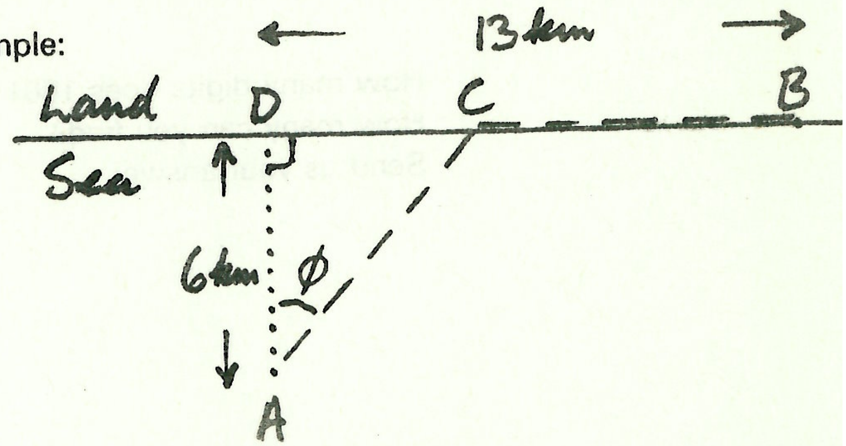
Alan Fekete.

Shortest time, without Calculus

Sir,

Some months ago I found a method for doing a type of question which is about 40 times faster than calculus.

Please look at the following example:



A man wants to go from A to B in shortest time. He can swim 5 km/h and run 10 km/h. Find the angle ϕ he must start off at.

By calculus, I should do it like this:

Assume CD = x km.

Time = $\sqrt{x^2+6^2}/5 + (13 - x)/10$

$dT/dx = x/5\sqrt{x^2 + 36} - 1/10$

T is a minimum, so $x/5\sqrt{x^2 + 36} - 1/10 = 0$

$\therefore 4x^2 = x^2 + 36$

$\therefore x = \sqrt{12}$

$\therefore \phi = \tan^{-1}(\sqrt{12}/6) = 30°$

By my new method, I do it like this:

$\phi = \sin^{-1}$(Velocity in sea/Velocity on land)

$= \sin^{-1}5/10 = 30°$

I read in a Physics book that all waves travel not by the shortest route, but by the easiest route (taking the shortest time). So this question becomes: A wave can travel 5 km/h in the medium "sea" and 10 km/h in the medium "land". And it becomes a question of finding the critical angle.

I wonder if anybody found this method before me? If not, I hope everybody can learn this method.

Kit Lam,
Year 12,
Trinity Grammar School

Editor's comments: This is an excellent letter. Kit has shown us how we might apply a principle learnt in Science to a problem met with in Mathematics. We must try to follow his example, and not compartmentalise our knowledge, but take a broad view. I must say that I have not previously seen this method of solution.

Incidentally, in a forthcoming issue we may see how Snell's Law of Refraction follows from the more general Fermat's Principle of Least Time.

Exact value of tan 1°

Sir,

I have been investigating the possibilities of finding the exact value of tan 1°. By using the formula $\tan 2A = 2\tan A/1 - \tan^2 A$, I have found values, in relation to tan 1°, which I called x, for tan 2°, tan 4°, tan 8° and tan 16°. I then related tan 16° to tan 15° $= (2 - \sqrt{3})$ and tan 1° = x, using the formula $\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$. Thus, I finished with an equation with one unknown in x^{17}, with every second coefficient being irrational.

Question: Can the exact values of the roots of this equation, of which one is tan 1°, be ascertained?

Philip Stott,
Year 11,
Newington College.

Editor's Comments: Indeed $x = \tan 1°$ satisfies a polynomial with integer coefficients, and so is an "algebraic number". To see this, we must first develop some formulas:

We start with

(1) $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$.

Putting A for B, we obtain

(2) $\tan 2A = 2 \tan A/(1 - \tan^2 A)$.

Putting 2A for B we obtain

(3) $\tan 3A = (\tan A + \tan 2A)/(1 - \tan A \tan 2A)$

which, in view of equation (2), becomes

$$\tan 3A = (3 \tan A - \tan^3 A)/(1 - 3 \tan^2 A).$$

Proceeding in the same way, we can develop a formula for tan (nA) in terms of tan A. This formula involves the binomial coefficients, found in Pascal's triangle. (These numbers pop up everywhere — see M. Reynold's article "Sequences generated by polynomials", page 3).

Indeed,

$$\tan(nA) = ({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - + \ldots) / (1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - + \ldots).$$

Now let $A = 1°$, $x = \tan A$, and $n = 45$;

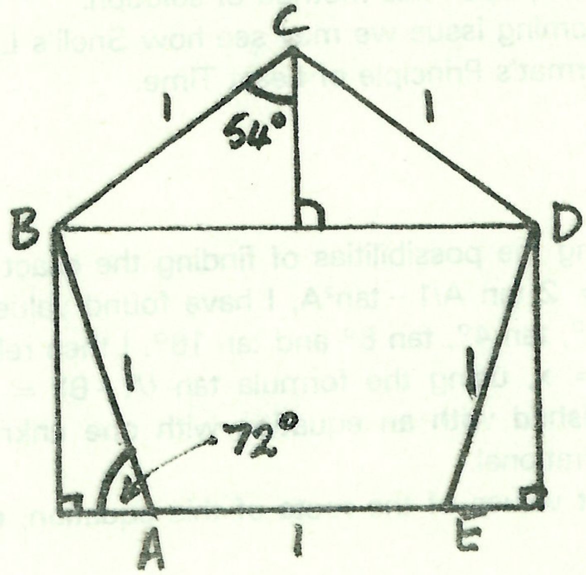
$\tan 45° = 1$, so we obtain

$$1 - {}^{45}C_1 x - {}^{45}C_2 x^2 + {}^{45}C_3 x^3 + {}^{45}C_4 x^4 - - + + \ldots = 0,$$

a polynomial with integer coefficients satisfied by $x = \tan 1°$.

Indeed the 45 roots of this equation are tan 1°, tan 5°, tan 9°, . . . , tan 177°.

Now we consider Philip's question, whether we can find tan 1° exactly.



From the figure, which shows a regular pentagon, we can see that

$2\sin 54^\circ = 1 + 2\cos 72^\circ$
$= 1 - 2\cos 108^\circ$
$= 1 - 2(1 - 2\sin^2 54^\circ)$.
$\therefore 4\sin^2 54^\circ - 2\sin 54^\circ - 1 = 0$.
$\therefore \sin 54^\circ = (1 + \sqrt{5})/4$.
$\therefore \cos 108^\circ = 1 - 2\sin^2 54^\circ = (1 - \sqrt{5})/4$.
$\therefore \cos 72^\circ = -\cos 108^\circ = (\sqrt{5} - 1)/4$.
$\therefore \sin 18^\circ = \cos 72^\circ = (\sqrt{5} - 1)/4$.
$\therefore \cos 18^\circ = \sqrt{(1 - \sin^2 18^\circ)} = \sqrt{((5 + \sqrt{5})/8)}$.
$\therefore \sin 9^\circ = \sqrt{((1 - \cos 18^\circ)/2)}$
$= \sqrt{((1 - \sqrt{((5 + \sqrt{5})/8)})/2)}$
and $\cos 9^\circ = \sqrt{((1 + \cos 18^\circ)/2)}$
$= \sqrt{((1 + \sqrt{((5 + \sqrt{5})/8)})/2)}$

Now, we can find $\tan 1^\circ$ if we can solve cubic equations, since

$3\sin 3^\circ - 4\sin^3 3^\circ = \sin 9^\circ$
$4\cos^3 3^\circ - 3\cos 3^\circ = \cos 9^\circ$,
and then
$3\sin 1^\circ - 4\sin^3 1^\circ = \sin 3^\circ$
$4\cos^3 1^\circ - 3\cos 1^\circ = \cos 3^\circ$,
and finally, $\tan 1^\circ = \sin 1^\circ/\cos 1^\circ$.

I believe it **is** possible to find the solutions of the above cubic equations in terms of surds, and thus find $\tan 1^\circ$, but I am going to have to do some research before I can complete this answer. Indeed, it may lead me to write an article in a future issue of Parabola.

Jeopardy

In response to Philip Stott's letter of the last issue, I would like to show that, as we might hope, A does have a better chance of winning. Suppose A bet X of his 500 points on the last question, and B bet Y of his 400 points, and suppose

$$\Pr(\text{A is correct}) = \Pr(\text{B is correct}) = p.$$

Then

$$\begin{aligned}\Pr(\text{A wins}) &= p^2.\Pr(500 + X > 400 + Y) \\ &\quad + p(1-p).\Pr(500 + X > 400 - Y) \\ &\quad + (1-p)p.\Pr(500 - X > 400 + Y) \\ &\quad + (1-p)^2.\Pr(500 - X > 400 - Y) \\ &= p^2\,((100 + X)/400) + p(1 - p).1 + (1 - p).p((100 - X)/400) \\ &\quad + (1 - p)^2\,((500 - X)/400),\end{aligned}$$

where $((100 + X)/400)$ only applies for $X \leq 300$,
and $((100 - X)/400)$ only applies for $X \leq 100$.

If the last quiz-question is very easy, then p $\cong$ 1, and thus

$$\Pr(A \text{ wins}) \cong ((100 + X)/400).$$

So A will bet more than 300 points and almost be sure of winning.

If however, the last question is very hard, then p $\cong$ 0, and $1-p \cong 1$, so that

$$\Pr(A \text{ wins}) \cong ((500 - X)/400),$$

so A will bet less than 100 points and almost be sure of winning.

So, as Philip pointed out, a confident A would bet 300 and feel sure of winning, but as he failed to mention, a very unconfident A would bet 100 and feel sure of winning.

So, as might have been expected, A has a better chance of winning than B.

David Dowe,
Year 12,
Geelong Grammar

Editor's comment: David's is the only response I have received so far. Any further comments?