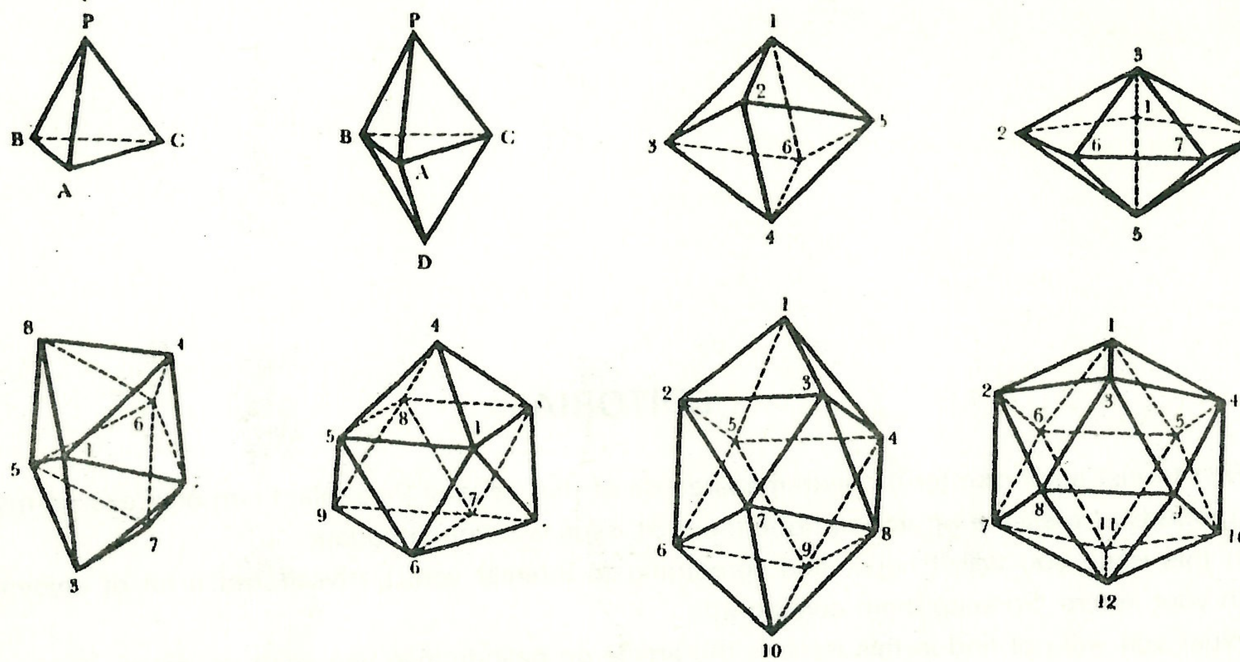


POLYHEDRA
II: THE DELTAHEDRA

A.V. Nikov*

The marvellous formula of Euler,

$$V - E + F = 2$$

can yield interesting results other than that concerning the Platonic Solids.

Suppose we ask the following question. How many convex polyhedra are there with only equilateral triangles for faces (but not necessarily the same number of faces meeting at each vertex)? These polyhedra are called deltahedra, after the Greek letter delta Δ. (Here "convex" is to be taken to mean "strictly convex", in the sense that no two adjacent faces should even lie in the same plane).

Every face has three edges, so, counting each edge twice, we have

$$2E = 3F \qquad \text{(i)}$$

At each vertex, either 3, 4 or 5 edges meet, since if 6 edges met at a vertex, the polyhedron would flatten out at that vertex, which we don't allow.

Consequently, let V_3, V_4, V_5 be the number of vertices where 3, 4, 5 edges meet, respectively.

**Mrs Nikov is a Senior Tutor in Pure Mathematics at the University of New South Wales. Her article POLYHEDRA 1: THE PLATONIC SOLIDS appeared in Parabola Vol. 14. No. 1.*

We note that

$$V_3 + V_4 + V_5 = V \qquad \text{(ii)}$$

and, counting every edge twice (once for each end),

$$3V_3 + 4V_4 + 5V_5 = 2E \qquad \text{(iii)}$$

Now, $V - E + F = 2,$

so $6V - 6E + 6F = 12,$

so by (i) $6V - 6E + 4E = 12,$

or, $6V - 2E = 12,$

so by (ii), (iii) $6(V_3 + V_4 + V_5) - (3V_3 + 4V_4 + 5V_5) = 12,$

so finally, $3V_3 + 2V_4 + V_5 = 12. \qquad \text{(iv)}$

This last equation has only finitely many solutions with V_3, V_4, V_5 non-negative, and we can tabulate them systematically, as follows:

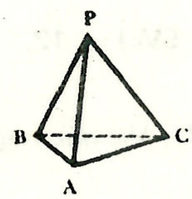
Clearly $V_3 \leqslant 4$. For each value of V_3, $V_3 = 0, 1, 2, 3, 4$, we write $2V_4 + V_5 = 12 - 3V_3$. Thus $V_4 \leqslant \frac{1}{2}(12 - 3V_3)$. For each such value of V_4, V_5 is determined by $V_5 = 12 - 3V_3 - 2V_4$.

Thus we arrive at the following table of solutions:

	V_3	V_4	V_5
1.	0	0	12
2.	0	1	10
3.	0	2	8
4.	0	3	6
5.	0	4	4
6.	0	5	2
7.	0	6	0
8.	1	0	9
9.	1	1	7
10.	1	2	5
11.	1	3	3
12.	1	4	1
13.	2	0	6
14.	2	1	4
15.	2	2	2
16.	2	3	0
17.	3	0	3
18.	3	1	1
19.	4	0	0

This table lists 19 solutions to (iv), but there is no guarantee that we can actually construct a polyhedron to each set of specifications. Indeed, using geometric arguments, we shall show that only 8 of them are constructible.

Consider those cases for which $V_3 > 0$, that is, there is at least one vertex where only 3 edges meet. (This vertex is quite "sharp", and it turns out to be quite hard to construct a **convex** polyhedron with such a vertex). If we join three triangles to form the vertex, then we can either (a) close it off with a fourth triangle to form a tetrahedron or (b) attach triangles around the base to form the "double triangular pyramid" (see figures 1(a), (b).)



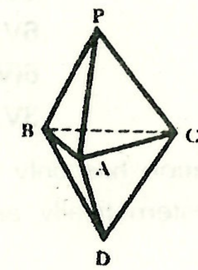


Figure 1

Indeed, these are the only two deltahedra with a triple vertex. For if we try to close off the triple vertex in any other way, we must insert a triangle at A, B or C (see figure 2) but in doing so we push two triangles apart, causing the sides to flatten out (which is not permitted).

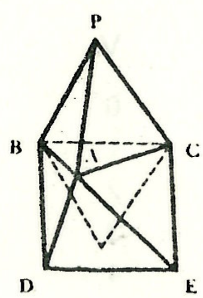
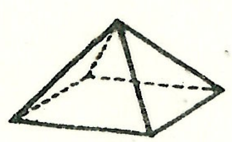


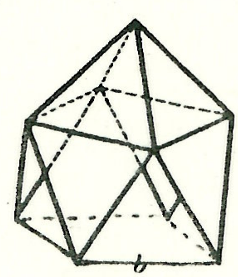
Figure 2

Thus we have shown that of the cases 8-19, only two are constructible, namely (19) and (16). The remaining cases 1-7 are constructible, with the exception of No. 2 ($V_3 = 0$, $V_4 = 1$, $V_5 = 10$). To show that No. 2 is not constructible, we shall try to form it.

Start with four triangles joined to form the one vertex of order 4 (see figure 3 (a)). All other vertices have order 5, so we see that the construction must proceed as in figure 3(b). We now have 9 vertices all together, and it is easy to see that it is impossible to have two more vertices.



a



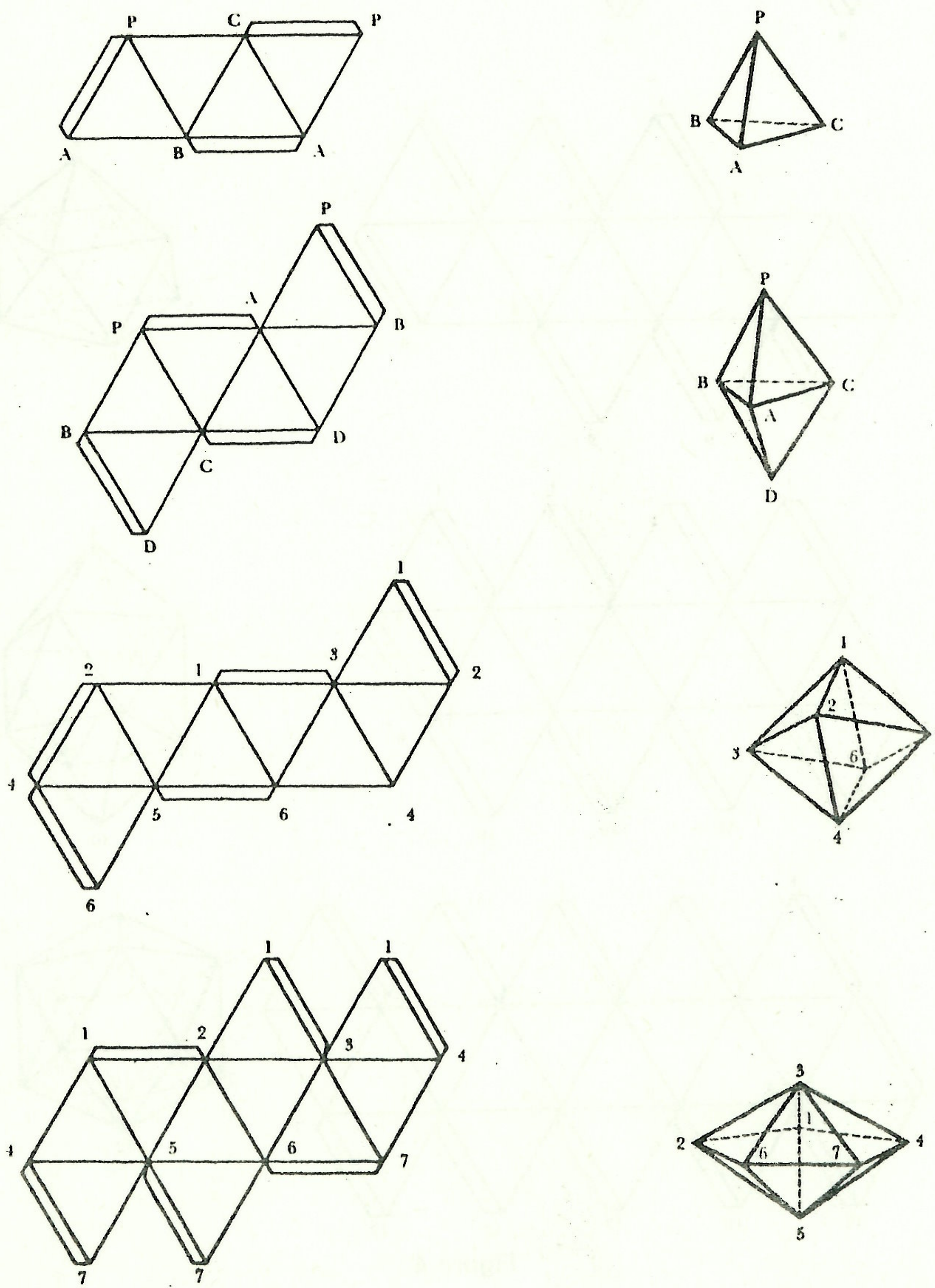
b

Figure 3

To show that the remaining cases are indeed constructible we give you patterns for them in figure 4.

You may be interested to know that the apparently simple theorem which states "There are precisely eight deltahedra" was proved only as recently as 1947, by two famous mathematicians, B.L. van der Waerden and H. Freudenthal.

Problem. How many types of polyhedra are there with precisely 5 (not necessarily regular) faces?



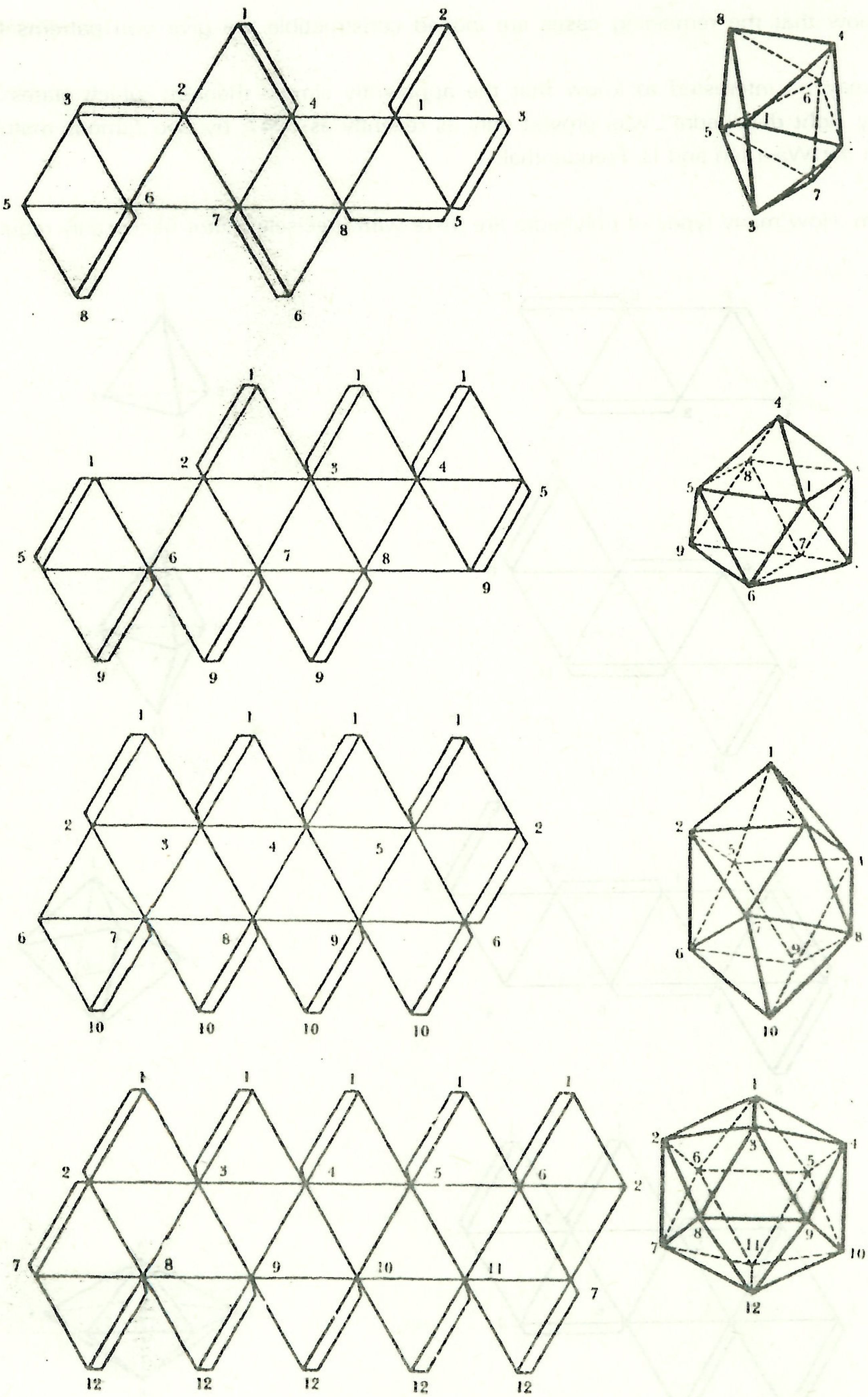


Figure 4