HOW TO AVOID COMMON MATHEMATICAL ERRORS

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To solve a mathematical problem does not just mean to perform a certain number of manipulations. The most important thing is that the solution be complete.

Check Solutions

It is vitally important that the student be able at every stage in a solution to realize what has been done and what there is left to do. Consider some details. Some students are not familiar with the fact that squaring both sides of an equation can give rise to extraneous roots. Most students can of course simplify an equation by means of clear-cut manipulations, but it is far from being the case that every student is capable of realizing that a solution has been lost or acquired, and many don't even give thought to such things.

How to solve an equation? The best method is clearly each time to replace the given equation with an **equivalent** equation. In practice, however, this ideal situation is rare. As a rule, an equation is replaced by another equation that is a consequence of, but not necessarily equivalent to, the first. In this case there is no loss of roots, but extraneous roots may appear and therefore a check is necessary. Take, for example, the equation $\ln(2x+1) = \ln x$. A consequence of this equation is the equation $2x+1 = x$ which has the solution $x = -1$, but clearly $x = -1$ is not a solution of the original equation.

It must be stressed that it is not permissible to replace a given equation by one which is not a consequence of the first. In this case a root can be lost for good. Consider the following example. If the equation $\log_2 x^2 = 0$ is replaced by $2 \log_2 x = 0$ we shall come to the conclusion that $x = 1$ and the solution $x = -1$ of the original equation is lost.

Inequalities

A great many mistakes are made in the solution of inequalities. It is a matter of wonder that so many mistakes are made by students when solving the simplest kind of inequality. Apparently this is due to an improperly understood analogy between equations and inequalities. The reasoning goes roughly like this: "Since the solution of the equation $-x-x^3 = 0$ is $x = 0$ the solution of the inequality $-x-x^3 > 0$ is $x > 0$".

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Logarithms

Another mistake is that some students forget that the logarithmic function is not defined for all values of x. For example, when solving the inequality $\log_2 x < 1$ they reason as follows: "We rewrite the inequality as $\log_2 x < \log_2 2$ and therefore $x < 2$". The correct answer is $0 < x < 2$.

Graphs

The ability to represent by means of graphs functional relationships given by formulae is particularly important. That is why examination papers contain problems involving the graphing of functions.

Consider an example. "Draw the graph of the function $y = \log_2(-x)$". Sometimes a student gives an answer like this: "The graph of the function does not exist since negative numbers do not have logarithms". The mistake here is the failure to grasp the fact that $(-x)$ does not by any means always represent a negative number.

Let us recall definitions which we need to know:

Definition 1. $\sqrt{a}$ = the nonnegative root of the equation $x^2 = a$ $(a \geqslant 0)$

Definition 2. $|a| = \begin{cases} a, & a \geqslant 0 \\ -a, & a < 0 \end{cases}$

Definition 3. $\log_a b$ = the root of the equation $a^x = b$, $a > 0$, $a \neq 1$, $b > 0$. Note that since $|a|$ takes values a or $-a$ we must have $|a|^2 = a^2$ and clearly $|a| \geqslant 0$. Thus $|a|$ is the nonnegative root of the equation $x^2 = a^2$ and therefore

$$\sqrt{a^2} = |a| \qquad (1)$$

We also note that $\log_a x^2 = \log_a |x|^2 = 2 \log_a |x|$, i.e.

$$\log_a x^2 = 2 \log_a |x| \qquad (2)$$

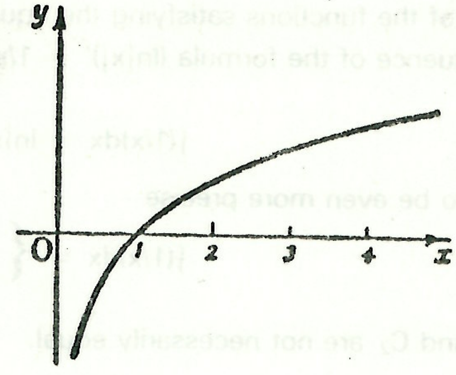
Many mistakes are made by students who use the (wrong!) formulae $\sqrt{a^2} = a$, $\log_a x^2 = 2 \log_a x$ instead of (1) and (2). Here is a typical example:

" $\sqrt{(1-\cos^2 x)(1+\tan^2 x)} = \sqrt{\sin^2 x/\cos^2 x} = \sqrt{\tan^2 x} = \tan x$ "

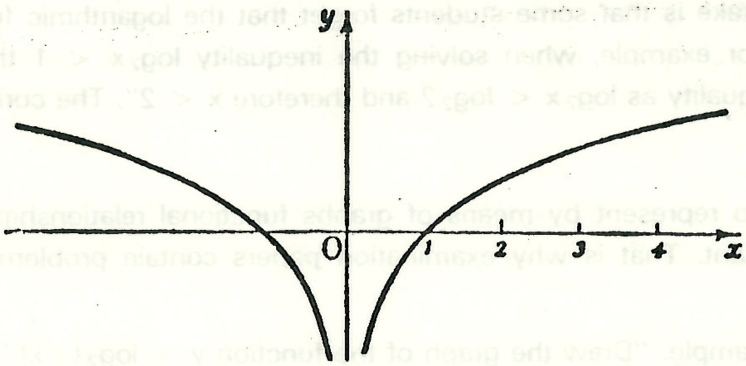
This answer is wrong, since $\sqrt{\tan^2 x} = |\tan x|$, not tan x. Here is another example:

"Since $\ln x^2 = 2 \ln x$ the graph of

the function $y = \frac{1}{2} \ln x^2$ is



The correct answer is "$y = \frac{1}{2} \ln x^2 = \frac{1}{2}.2 \ln |x| = \ln|x|$ and therefore the graph has the form



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Finding the Inverse of a Function

Many students are unable to find the inverse of a given function. The best they can do is to interchange x and y in the equation of the function. Let us illustrate one of the methods of finding the inverse of a function.

Suppose $f(x) = 2x - 1$;

For any $-\infty < t < \infty$, f sends t to $2t+1$, therefore f^{-1} must send $2t+1$ to t

i.e. $f^{-1}(2t+1) = t$.

Denote $2t+1$ by x. Then we have $t = \frac{1}{2}(x-1)$ and therefore $f^{-1}(x) = \frac{1}{2}(x-1)$.

Or another example: Let $f(x) = 2^x$, we have $f^{-1}(2^t) = t$ and if we denote 2^t by x, then "t" is the root of the equation $2^t = x$ $(x > 0)$. By definition 3 such a root is denoted by the symbol $\log_2 x$, therefore, $f^{-1}(x) = \log_2 x$.

Integrating 1/x

Now consider the following problem:

Find a function f(x) such that $f'(x) = 1/x$.

The common error in the solution of this problem is the following argument.

"Since $(\ln x)' = 1/x$ we can take $\ln x$ as a suitable f(x)". Where is the mistake?

Firstly the formula $(\ln x)' = 1/x$ is wrong, since the functions $(\ln x)'$ and $1/x$ have different domains and therefore cannot be equal. We can only claim that $(\ln x)' = 1/x$ provided the right hand side function has the restricted domain $x > 0$. Let us note that for $x < 0$, $[\ln(-x)]' = (1/(-x)).(-1) = 1/x$. Thus we have proved that

$$(\ln |x|)' = 1/x$$

and as one of the functions satisfying the equation $f'(x) = 1/x$ we can take $f(x) = \ln|x|$.

A consequence of the formula $(\ln|x|)' = 1/x$ is the relation

$$\int (1/x)dx = \ln|x| + C, \text{ not } \ln x + C$$

Remark To be even more precise

$$\int (1/x)dx = \begin{cases} C_1 + \ln x, & x > 0 \\ C_2 + \ln(-x), & x < 0 \end{cases}$$

where C_1 and C_2 are not necessarily equal.

Finally, we shall give some "solutions" of problems and urge you to send us the reason why each of the solutions is wrong.

Problem 1. Solve the equation $\tan 5x = \tan 3x$.
"The tangents of two angles are equal if and only if the difference of the angles is equal to an integral multiple of π. Hence $2x = \pi n$ and $x = \frac{1}{2}\pi n$ $(n = 0, \pm 1, \ldots)$".

Problem 2. Solve the equation $2 \sin x + \cos x = -1$.
"Let $t = \tan \frac{1}{2}x$, then $\sin x = (2t)/(1+t^2)$, $\cos x = (1-t^2)/(1+t^2)$ and hence $(4t)/(1+t^2) + (1-t^2)/(1+t^2) = -1$ or $4t + 1 - t^2 = -t^2 - 1$ and therefore $t = \frac{1}{2}$. Thus $\tan \frac{1}{2}x = -\frac{1}{2}$, $\frac{1}{2}x = -\tan^{-1}\frac{1}{2} + \pi n$, $x = -2\tan^{-1}\frac{1}{2} + 2\pi n$ $(n = 0, \pm 1, \ldots)$"

Problem 3. Find the equation of the line which passes through the intersection of the lines $x + y - 2 = 0$, $2x + y + 1 = 0$ and the point $(0, -1)$.
"We have $x + y - 1 + \lambda(2x + y + 1) = 0$ and since the point $(0, -1)$ belongs to this line, $-3 + 0.\lambda = 0$. But this is impossible. Hence there is no solution".

Problem 4. Find $\int \tan x \, dx$.

$$\text{"} \int \tan x \, dx = \int (\sec x \tan x)/(\sec x) \, dx = \log(\sec x) + C. \text{"}$$

Problem 5. Simplify $\sqrt{(1 + \sin 2x)}$.

$$\text{"} \sqrt{(1 + \sin 2x)} = \sqrt{(\sin^2 x + \cos^2 x + 2 \sin x \cos x)} = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x \text{"}.$$

Problem 6. Find the area between the curve $y = 1/x$ and the lines $x = -3$, $x = -1$, and $y = 0$.

$$\text{"Area} = \int_{-3}^{-1} (1/x)\, dx = \ln x \Big|_{-3}^{-1}$$
$$= \ln(-1) - \ln(-3)$$
$$= \ln((-1)/(-3)) = \ln \tfrac{1}{3} = -\ln 3\text{"}.$$

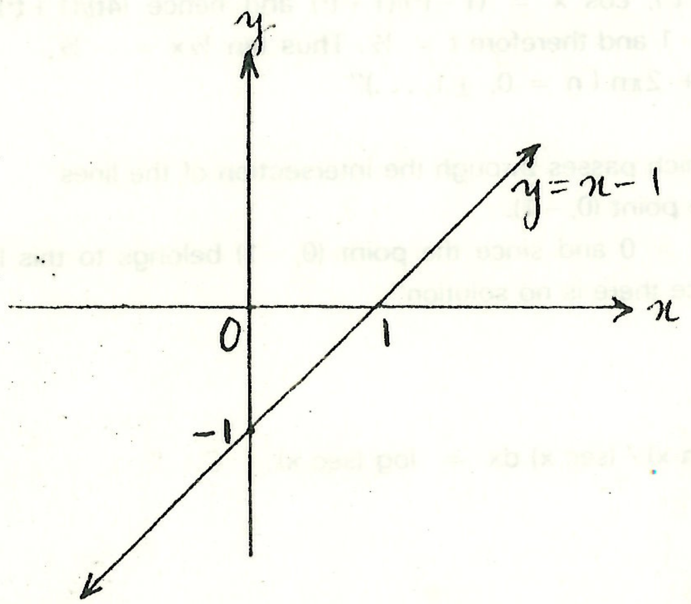
Problem 7. Solve the inequality $\log_x (x+1) > 1$.
"It follows from $\log_x(x+1) > 1$ that $x+1 > x$ and therefore any x is a solution".

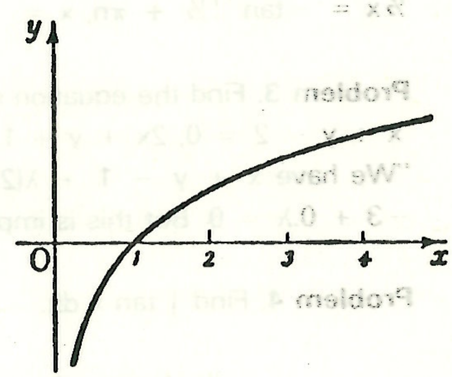
Problem 8. Solve the inequality $(2x+1)^2 > (3x-1)^2$.
"By taking the square roots we have $2x+1 > 3x-1$ or $-x > -2$, or $x < 2$".

Problem 9. Draw the graph of the function $y = \sqrt{(x^2 - 2x + 1)}$.
"$y = \sqrt{(x^2 - 2x + 1)} = \sqrt{(x-1)^2} = x - 1$.

Therefore the graph is that shown in the figure below on the left."

Problem 10. Draw the graph of the function $y = \ln x^4 - (3/2) \ln x^2$.
"$y = 4 \ln x - (3/2) \times 2 \ln x = \ln x$. Hence the graph is that shown in the figure on the right."





Problem 11. Find $\int (1/(ax+b))\, dx$ $(a \neq 0)$.
"Substitute $ax + b = t$, then $x = (t-b)/a$, $dx = (1/a)\, dt$ and therefore
$\int (1/(ax+b)) = \int (1/a) \times (1/t).dt = (1/a) \ln t + C = (1/a) \ln (ax - b) + C$".

Problem 12. Solve the inequality $|x+1| > -1$.
"By squaring the left and right hand side we have $x^2 + 2x + 1$, $x(x+2) > 0$. Thus $x > 0$ or $x < -2$".

Problem 13. Solve $3^x.8^{x/(x+2)} = 6$.
"$3^x.2^{3x/(x+2)} = 3^1\, 2^1$. Thus $x = 1$, $(3x)/(x+2) = 1$ and therefore $x = 1$".

Problem 14. If $y = x \ln x^2$, for what values of x is $dy/dx = 0$?
"$y = x.\, 2 \ln x = 2x \ln x$. Therefore

$$dy/dx = 2 \ln x + 2x \times (1/x) = 2(\ln x + 1)$$

Hence $\ln x = -1$ or $x = e^{-1}$".