LETTERS TO THE EDITOR

More on tan 1°

Sir,

Concerning the exact value of tan 1° (Letter to the Editor, Vol. 14 No. 1), in your comments you gave cubic equations for finding sin 3°, cos 3°. These are not necessary, since sin 18°, cos 18°, sin 15°, cos 15° are known as surds, and 3° = 18° − 15°. The cubic equations are not simple to solve, because cube roots of complex numbers are involved. I shall continue work on this.

Geoffrey J. Chappell,
Year 12, Kepnock High,
Bundaberg, Queensland.

Editor's comments: Geoffrey is quite right! We saw that

$$\sin 18^\circ = (\sqrt{5}-1)/4, \quad \cos 18^\circ = \sqrt{((5+\sqrt{5})/8)},$$

while $\cos 30^\circ = \sqrt{3}/2$, so

$$\sin 15^\circ = \sqrt{(1 - \cos 30^\circ)/2)} = \sqrt{((1-\sqrt{3}/2)/2)},$$
$$\cos 15^\circ = \sqrt{((1+\cos 30^\circ)/2)} = \sqrt{((1+\sqrt{3}/2)/2)},$$

so

$$\sin 3^\circ = \sin(18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ$$

and

$$\cos 3^\circ = \cos(18^\circ - 15^\circ) = \cos 18^\circ \cos 15^\circ + \sin 18^\circ \sin 15^\circ$$

can both be written in terms of surds. But then it seems that we still have to solve cubic equations to find sin 1°, cos 1°, and I have it on good authority that the solutions of these cubics necessarily involve surds of complex numbers.

Solutions to $x^2 + y^2 = z^2$

Sir,

Could you find for me some information on how many solutions in positive integers there are to the equations $x^2 + y^2 = z^2$ for any particular x, y or z?

Geoffrey J. Chappell

Editor's comments: This is certainly an interesting question, but I think it is difficult to find the answer. I certainly don't know the answer. Here are just a few simple observations.

(i) It is not hard to see that if we fix any one of x, y or z, there are only finitely many solutions. This is obvious if z is fixed, since $0 < x \leqslant z$, $0 < y \leqslant z$. If we fix x, we have

$$z^2 - y^2 = x^2$$

or, $z + y = x^2/(z - y)$

so $z + y \leqslant x^2$,

so $y < \frac{1}{2}x^2$

so $z < x^2 + (\frac{1}{2}x^2)^2$.

[Similarly, if y is fixed, $x < \frac{1}{2}y^2$ and $z < y^2 + (\frac{1}{2}y^2)^2$.]

(ii) The next aspect of Geoffrey's question we might examine is, for which x or z are there no solutions?

There are no solutions if $x = 1$ or 2, but there is at least one solution for every other value of x. (Try to prove these statements!)

There are many values of z for which there are no solutions, for example, $z = 1, 3, 4, 6, 7, 9, \ldots$ (There are infinitely many such values of z.)

(iii) If we fix x, and for simplicity suppose x is odd, and suppose that y, z are **relatively prime**, then from

$$(z + y)(z - y) = x^2$$

we can deduce that

$$z + y = a^2, \quad z - y = b^2,$$

where $ab = x$, $a > b$, and a and b are relatively prime, and then

$$z = \frac{1}{2}(a^2 + b^2), \quad y = \frac{1}{2}(a^2 - b^2).$$

The number of such solutions is thus equal to the number of ways of writing x as a product of two different, relatively prime factors.

Thus, for example, if $x = 3$, there is only one such way, namely

$$3 = 3 \times 1, \quad a = 3, \quad b = 1, \quad z = 5, \quad y = 4,$$

while if $x = 15$, there are two such ways, namely

$$15 = 5 \times 3, \quad a = 5, \quad b = 3, \quad z = 17, \quad y = 8,$$

and

$$15 = 15 \times 1, \quad a = 15, \quad b = 1, \quad z = 113, \quad y = 112.$$

If x is even, the situation is somewhat more complicated. Why don't you, reader, examine this case, and send me your findings? I hope to comment further on Geoffrey's question in a later issue.

A Remarkable Continued Fraction

Sir,

At the National Mathematics Summer School in January this year, you taught me and a few others a method of obtaining the continued fraction of a number.

For example,

$$\sqrt{2} = 1 + 1/(2 + 1/(2 + 1/(2 + \ldots)))$$

From now on, I shall write

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots))$$

as $[a_0, a_1, a_2, \ldots]$.

I took the number $e = 2.718281828459\ldots$ and found that its continued fraction is

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots]$$

but here the pattern stopped, as the accuracy of my calculator restricted results.

Intrigued by this I tried e^2.

$$e^2 = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 1 \ldots],$$

and it seemed there was no pattern. A little discouraged, I tried $e^{1/2}$.

$$e^{1/2} = [1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, \ldots]$$

$$e^{1/3} = [1, 2, 1, 1, 8, 1, 1, 14, 1, 1, 20, \ldots]$$

$$e^{1/4} = [1, 3, 1, 1, 11, 1, 1, 19, 1, \ldots]$$

$$e^{1/5} = [1, 4, 1, 1, 14, 1, 1, 24, \ldots]$$

$$e^{1/6} = [1, 5, 1, 1, 17, 1, 1, 29, \ldots]$$

All this is very exciting! I conjecture that if x is a positive integer,

$$e^{1/x} = [1, (x-1), 1, 1, (3x-1), 1, 1, (5x-1), 1, 1, (7x-1), \ldots].$$

I would like to know if any mathematician found this result before me, if it can be proved or disproved, or anything you can tell me about it.

Rick Middleton,
Year 12 student,
Buttaba. N.S.W.

Editor's comments: First a few words about continued fractions. Given a number $x > 0$, we can write

$$x = a_0 + \epsilon_1,$$

where a_0 is an integer, and $0 \leqslant \epsilon_1 < 1$. If $\epsilon_1 > 0$, we can write

$$1/\epsilon_1 = a_1 + \epsilon_2, \text{ or } \epsilon_1 = 1/(a_1 + \epsilon_2),$$

where a_1 is an integer, and $0 \leqslant \epsilon_2 < 1$, and so on. Proceeding in this manner, we obtain

$$x = a_0 + 1/(a_1 + 1/(a_2 + \ldots)).$$

This is the continued fraction for x. The continued fraction terminates if and only if x is rational, and becomes periodic if and only if x is a quadratic irrational.

Thus for example, if $x = 13/8$,

$$\begin{aligned} x &= 1 + 5/8 \\ &= 1 + 1/(8/5) \\ &= 1 + 1/(1 + 3/5) \\ &= 1 + 1/(1 + 1/(5/3)) \\ &= 1 + 1/(1 + 1/(1 + 2/3)) \\ &= 1 + 1/(1 + 1/(1 + 1/(1 + 1/2))), \end{aligned}$$

while if $x = \frac{1}{2}(\sqrt{5} + 1)$

$$\begin{aligned} x &= 1 + \tfrac{1}{2}(\sqrt{5} - 1) \\ &= 1 + 1/(\tfrac{1}{2}(\sqrt{5} + 1)) \\ &= 1 + 1/(1 + 1/(1 + 1/(1 + \ldots))), \end{aligned}$$

which is periodic.

You may like to check that the continued fraction for $\sqrt{2}$ is, as Rick says,

$$1 + 1/(2 + 1/(2 + \ldots))).$$

Now we turn to Rick's wonderful discovery, or rather, rediscovery. For it turns out that this continued fraction was found by Euler.

Lambert had shown that

$$(e^{1/x} + 1)/(e^{1/x} - 1) = [2x, 6x, 10x, 14x, \ldots]. \qquad \text{(L)}$$

Later, Euler showed that

$$[1, x-1, 1, 1, 3x-1, 1, \ldots, (2k-1)x-1, 1, 1] = \\ = \frac{[2x, 6x, \ldots, (4k-2)x] + 1}{[2x, 6x, \ldots, (4k-2)x] - 1}. \qquad \text{(E)}$$

Letting $k \to \infty$ in (E), and using (L), we obtain

$$\begin{aligned} [1, x-1, 1, 1, 3x-1, 1, 1, \ldots] &= \\ &= ((e^{1/x} + 1)/(e^{1/x} - 1) + 1)/((e^{1/x} + 1)/(e^{1/x} - 1) - 1) \\ &= e^{1/x}. \end{aligned}$$

Unfortunately, neither (L) nor (E) is easy to prove, so I do not give a proof here.

In conclusion, I would encourage you, Rick, and others, to keep on doing such "experimental" mathematics. After all, this is surely the way to make discoveries!

PROOF THAT 10 IS AN EVEN NUMBER

$$\begin{aligned} 10 = 9 - 6 + 7 &= \text{IX} - \text{SIX} + \text{SEVEN} \\ &= -\text{S} + \text{SEVEN} \\ &= \text{EVEN} \end{aligned}$$