Parabola Volume 14, Issue 2 (1978)

SCHOOL MATHEMATICS COMPETITION 1978

Junior Division

Question 1. (i) Find a set of four different prime numbers such that the sum of every three of them is also a prime number. (1 is not a prime.)

(ii) Prove that there is no set of five different prime numbers such that the sum of every three of them is also a prime number.

Answer: (i) There are many such sets. For example {5,7,17,19}, {7,11,13,23}, {7,13,17,23}, {19,23,29,31}.

Note that 3 cannot belong to such a set since the other three numbers are all either of the form $3x+1$ ("+1 modulo 3") or $3x-1$ ("−1 mod 3") and then the sum of some three of the four numbers is divisible by 3 (and is greater than 3), so is composite.

(ii) As shown in (i), 3 cannot belong to the set, so all five numbers are +1 mod 3 or −1 mod 3. At least three of the five have the same residue mod 3, and the sum of such a set is divisible by 3 (and is greater than 3), so is composite. ☆

Question 2. Show that for all positive integers n,

$$(n!)^{1/n} < ((n+1)!)^{1/(n+1)}$$

(Remember that $n! = 1 \times 2 \times 3 \times \ldots \times n$)

Answer: $1 \times 2 \times 3 \times \ldots \times n < (n+1)^n$ since each of the n terms on the left is less than $n+1$.
That is,

$$n! < (n+1)^n$$

Multiplying by $(n!)^n$, we obtain

$$(n!)^{n+1} < (n!)^n(n+1)^n = (n! \times (n+1))^n = ((n+1)!)^n,$$

so, taking the $n(n+1)$th root, we have $(n!)^{1/n} < ((n+1)!)^{1/(n+1)}$. ☆

Question 3. a, b, c, d and e are consecutive positive integers less than 10,000 such that

$a + b + c + d + e$ is the cube of an integer

and $b + c + d$ is the square of an integer.

Find a, b, c, d, e.

Answer: $a = 673$, $b = 674$, $c = 675$, $d = 676$, $e = 677$.

For, $a = c-2$, $b = c-1$, $d = c+1$, $e = c+2$,

so $a + b + c + d + e = 5c = x^3$

and $b + c + d = 3c = y^2$.

$c = x^3/5$ is an integer, so x is a multiple of 5.

Write $x = 5m$. Then $c = 25m^3$.

Now, $c < 10000$, so $25m^3 < 10000$,

so $m^3 < 400$

so $m \leqslant 7$.

For each value of $m \leqslant 7$, we can calculate $c = 25m^3$. Then we can investigate whether or not 3c is a square.

m	m^3	$c = 25m^3$	3c	square?	
1	1	25	75	no	
2	8	200	600	no	
3	27	675	2025	yes	$(2025 = 45^2)$
4	64	1600	4800	no	
5	125	3250	9750	no	
6	216	5400	16200	no	
7	343	8575	25725	no	

so $c = 675$, and the rest follow. ☆

Question 4. Three motorists A, B and C often travel on a certain highway, and each always travels at a constant speed, A the fastest, C the slowest. One day when the three travel in the same direction, B overtakes C, five minutes later A overtakes C, and in another three minutes A overtakes B. On another occasion when they again travel in the same direction, A overtakes B first, then nine minutes later, A overtakes C. How much later does B overtake C?

Answer: B overtakes C 15 minutes after A overtakes C.

For, let the speeds of A, B, C be s(A), s(B), s(C).

The information regarding the first day's trip tells us that

$$8(s(B) - s(C)) = 3(s(A) - s(C)),$$

or

$$(s(A) - s(C)) : (s(B) - s(C)) = 8:3.$$

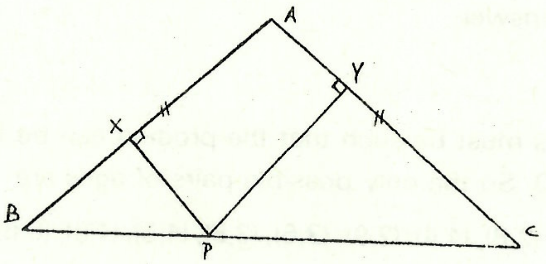
Suppose that on the second day B overtakes C t minutes after A overtakes C. Then

so $$(9 + t)(s(B) - s(C)) = 9(s(A) - s(C)),$$

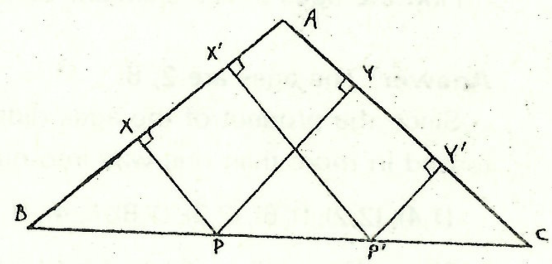
$$(9 + t):9 = (s(A) - s(C)):(s(B) - s(C)) = 8:3,$$

from which it follows that $t = 15$. ☆

Question 5. (i) ABC is an isosceles triangle as shown. Show that the sum of the distances PX and PY is the same for all points P on BC.



(i)



(ii)

(ii) Conversely, suppose ABC is a triangle, and that P and P′ are distinct points on BC such that

$$PX + PY = P'X' + P'Y'.$$

Show that AB = AC.

Answer: (i) △PXB is similar to △PYC

so $PX : PB = PY : PC = r$, say

so $PX = rPB$, $PY = rPC$

so $PX + PY = r(PB + PC) = rBC$.

Now rBC does not depend on the position of P, so PX + PY is the same for all points P on BC.

(ii) △PXB is similar to △P′X′B,

△PYC is similar to △P′Y′C.

so, $PX : PB = P'X' : P'B = r$, say,

and $PY : PC = P'Y' : P'C = s$, say.

So $PX = rPB$, $P'X' = rP'B$,

$PY = sPC$, $P'Y' = sP'C$.

Now we are told that

$$PX + PY = P'X' + P'Y',$$

so $rPB + sPC = rP'B + sP'C$,

or $r(P'B - PB) = s(PC - P'C)$,

$rPP' = sPP'$,

so $r = s$.

It follows that △PXB is similar to △PYC,

so $\angle ABC = \angle ACB$,

so △ABC is isosceles, with AB = AC. ☆

Senior Division

Question 1. A says to B, "Today is the birthday of my two children. See if you can guess their ages (they are both under 10). I'll give you a clue. I won't tell you the product of their ages, because that won't help you. However, the quotient of their ages is . . ." (B hears but we don't).

B says to A, "I still can't tell their ages."

A says "You're right. But if I had told you the difference between their ages instead of the quotient, you would have been able to work them out."

B says, "Oh, then their ages are . . ."

Find the ages of A's children. Give reasons for your answer.

Answer: The ages are 2, 8.

Since the product of the ages didn't help B, the ages must be such that the product can be factorised in more than one way into numbers less than 10. So the only possible pairs of ages are

(1,4), (2,2), (1,6), (2,3), (1,8), (2,4), (1,9), (3,3), (2,6), (3,4), (2,8), (4,4), (2,9), (3,6), (3,8), (4,6), (4,9), (6.6)

Since A says ". . .if I had told you the difference between their ages. . .you would have been able to work them out" the difference must be such that only one of the above pairs gives rise to that difference. Thus the ages are (2,6), (2,8), or (1,9).

Further, knowing the quotient of the ages didn't help B identify the ages amongst those in the original list, so the quotient must be such that at least two pairs give rise to that quotient.

But 2,6 have quotient 3, which no other pair have, and 1,9 have quotient 9, which no other pair have. So the ages are 2,8. ☆

Question 2. A sideshow has mechanical clowns which drop identical table-tennis balls onto boards with six alleys. Six balls are fed into a clown.

(i) How many different final arrangements of the balls can occur? (The boards are such that each ball must end up in one of the alleys.)

(ii) Which is the more likely of the following two final outcomes? (The balls are equally likely to end up in any one of the six alleys.)

(a) Some alley ends up with four or more balls.

(b) Each alley ends up with an even number of balls (0 is even). Give reasons for your answers.

Answer: (i) The number of arrangements is 462.

The possible arrangements are as follows:

6 balls in one alley; 5 balls in one alley, 1 in another, and so on.

The complete list is 6; 5,1; 4,2; 4,1,1; 3,3; 3,2,1; 3,1,1,1; 2,2,2; 2,2,1,1; 2,1,1,1,1; 1,1,1,1,1,1.

The arrangement labelled "6" can actually be obtained in 6 different ways — the 6 balls could be in any of the 6 alleys. The arrangement labelled "5,1" can be obtained in 30 different ways — the 5 balls could be in any of the 6 alleys, and the 1 could be in any of the remaining 5 alleys, yielding $6 \times 5 = 30$ different arrangements.

The complete list is

6	6	arrangements
5,1	30	
4,2	30	
4,1,1	60	
3,3	15	
3,2,1	120	
3,1,1,1	60	
2,2,2	20	
2,2,1,1	90	
2,1,1,1,1	30	
1,1,1,1,1,1	1	
The total is	462.	

(ii) (a) is more likely than (b).

The event described in (a) is the union of the (mutually exclusive) events 6; 5,1; 4,2; 4,1,1, while the event (b) is the union of the events 6; 4,2; 2,2,2, so

$$\Pr(a) = \Pr(6) + \Pr(5,1) + \Pr(4,2) + \Pr(4,1,1),$$
$$\Pr(b) = \Pr(6) + \Pr(4,2) + \Pr(2,2,2).$$

To compare these two probabilities, we need only calculate the probabilities of 5,1; 4,1,1; and 2,2,2.

Now, the probability of 5,1 is 30 × the probability of 5 in alley A, 1 in alley B, which event we shall denote A^5B^1. To calculate **this** probability, we note that the balls can drop in any of the 6 orders

A A A A A B
A A A A B A
A A A B A A
A A B A A A
A B A A A A
B A A A A A,

and each of these events has probability $1/6^6$.

So $\Pr(5,1) = 30 \times \Pr(A^5B^1) = 30 \times 6 \times 1/6^6 = 180/6^6$.

Similarly, $\Pr(4,1,1) = 60 \times \Pr(A^4B^1C^1) = 60 \times 30 \times 1/6^6 = 1800/6^6$,

$\Pr(2,2,2) = 20 \times \Pr(A^2B^2C^2) = 20 \times 90 \times 1/6^6 = 1800/6^6$

Since $\Pr(5,1) + \Pr(4,1,1) > \Pr(2,2,2)$,

$\Pr(a) > \Pr(b)$. ☆

Question 3. (i) Prove that there are infinitely many sets of five consecutive positive integers a, b, c, d, e such that

$a + b + c + d + e$ is the cube of an integer
and $b + c + d$ is the square of an integer.

(ii) Prove that there is no set of eight consecutive integers such that the sum of the first five is the cube of an integer and the sum of the last three is the square of an integer.

Answer: (i) Let $a = 675s^6 - 2$, $b = 675s^6 - 1$, $c = 675s^6$, $d = 675s^6 + 1$, $e = 675s^6 + 2$, where s is any integer.

Then $a + b + c + d + e = 3375s^6 = (15s^2)^3$,

and $b + c + d = 2025s^6 = (45s^3)^2$.

We have $a + b + c + d + e = 5c = x^3$,

$b + c + d = 3c = y^2$.

So $x = 5\ell$, $y = 3m$, $c = 25\ell^3 = 3m^2$.

So $\ell = 3n$, $m = 5p$, $9n^3 = p^2$.

So $n = s^2$, $p = 3s^2$, $\ell = 3s^2$, $m = 15s^3$, $c = 675s^6$.

(ii) Suppose the numbers are

$a-2, a-1, a, a+1, a+2, a+3, a+4, a+5.$

Then $5a = x^3$, $3a + 12 = y^2$,

Therefore $5|x$, $x = 5z$,

$3 \times 125z^3 = 5y^2 - 60$,

$y^2 = 75z^3 + 12$,

$y^2 \equiv 2 \bmod 5$

But this last congruence is impossible. ☆

Question 4. For which real numbers x is

$$\sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}}$$

an integer?

Answer: If n is any integer, and $x = \frac{1}{2}(n^3 + 3n)$, then $y = \sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}}$ is an integer (indeed $y = n$), and these are the only real numbers x for which y is an integer.

For, let $a = \sqrt[3]{x + \sqrt{x^2 + 1}}$, $b = \sqrt[3]{x - \sqrt{x^2 + 1}}$.

Then $a^3 = x + \sqrt{x^2 + 1}$, $b^3 = x - \sqrt{x^2 + 1}$,

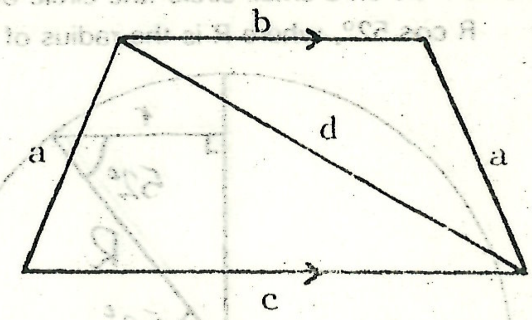
$a^3 + b^3 = 2x$, $a^3b^3 = -1$, $ab = -1$.

So $y^3 = (a + b)^3 = (a^3 + b^3) + 3ab(a + b) = 2x - 3y$,

or, $x = \frac{1}{2}(y^3 + 3y)$.

It follows that $y = n$ if and only if $x = \frac{1}{2}(n^3 + 3n)$. ☆

Question 5. (i) Given an isosceles trapezium, as shown in the diagram, prove that $d^2 = a^2 + bc$.



(ii) Hence or otherwise find the great circle distance between London (52°N, 0°E) and Sydney (35°S, 152°E), assuming the earth is a sphere of radius 6366 km. (The great circle distance between two points on a sphere is the shortest distance between them across the surface of the sphere. The shortest path is an arc of a great circle. A great circle is the intersection of the sphere's surface with a plane through the sphere's centre.)

Answer: (i) By the "cosine rule",

$$d^2 = a^2 + c^2 - 2ac\cos\theta$$

and $d^2 = a^2 + b^2 - 2ab\cos(180° - \theta)$

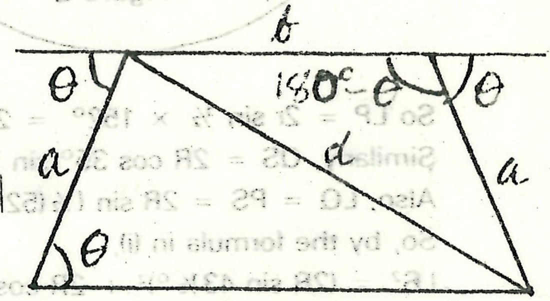
$$= a^2 + b^2 + 2ab\cos\theta.$$

So $\cos\theta = (a^2 + c^2 - d^2)/(2ac) = (d^2 - (a^2 + b^2))/(2ab)$.

So $(b + c)d^2 = b(a^2 + c^2) + c(a^2 + b^2)$

$$= (b + c)(a^2 + bc),$$

so $d^2 = a^2 + bc$.



(ii) The great circle distance between London and Sydney is 16,400 km. For the four points L(0°E,52°N), S(152°E,35°S), P(152°E,52°N), Q(0°E,35°S) form an isosceles trapezium (in a plane which slices through the earth) (see figure 1).

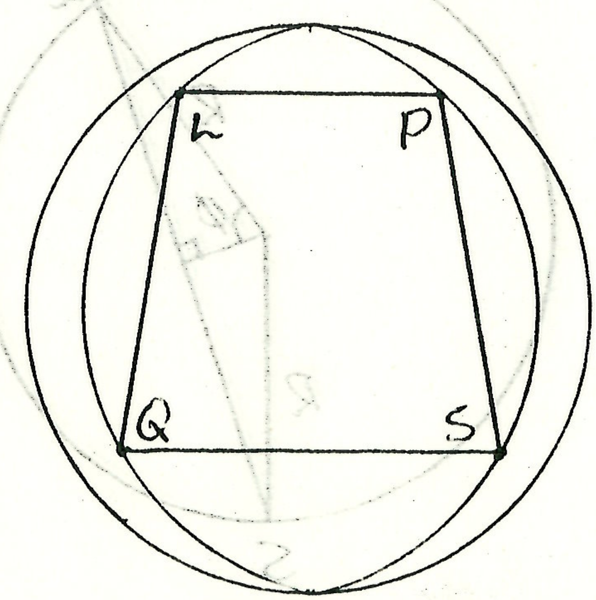


Figure 1

L and P lie on a small circle (the circle of latitude 52°N). The radius r of this small circle is $r = R \cos 52°$, where R is the radius of the earth (see figure 2).

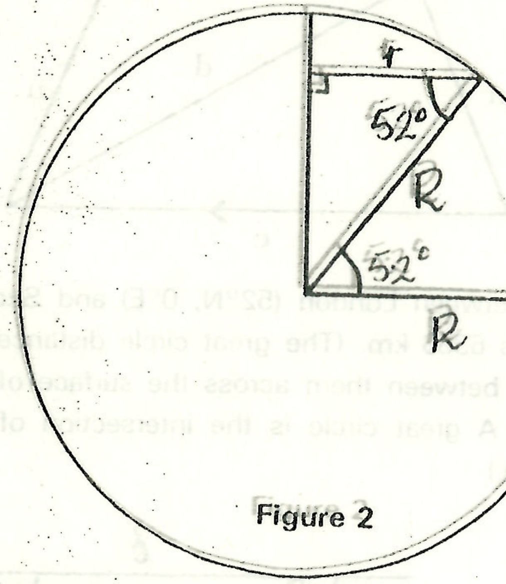


Figure 2

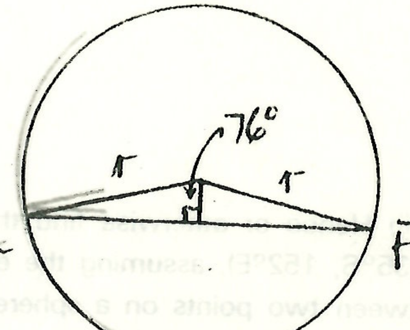


Figure 3

So $LP = 2r \sin \tfrac{1}{2} \times 152° = 2R \cos 52° \sin 76°$ (see figure 3).

Similarly, $QS = 2R \cos 35° \sin 76°$.

Also, $LQ = PS = 2R \sin (\tfrac{1}{2}(52° + 35°)) = 2R \sin 43\tfrac{1}{2}°$.

So, by the formula in (i),

$$LS^2 = (2R \sin 43\tfrac{1}{2}°)^2 + 2R \cos 52° \sin 76° \, . \, 2R \cos 35° \sin 76°$$
$$= 4R^2(\sin^2 43\tfrac{1}{2}° + \cos 52° \cos 35° \sin^2 76°),$$
$$LS = 2R \sqrt{(\sin^2 43\tfrac{1}{2}° + \cos 52° \cos 35° \sin^2 76°)}.$$

Now, LS as just calculated is the straight-line distance between London and Sydney through the earth (see figure 4).

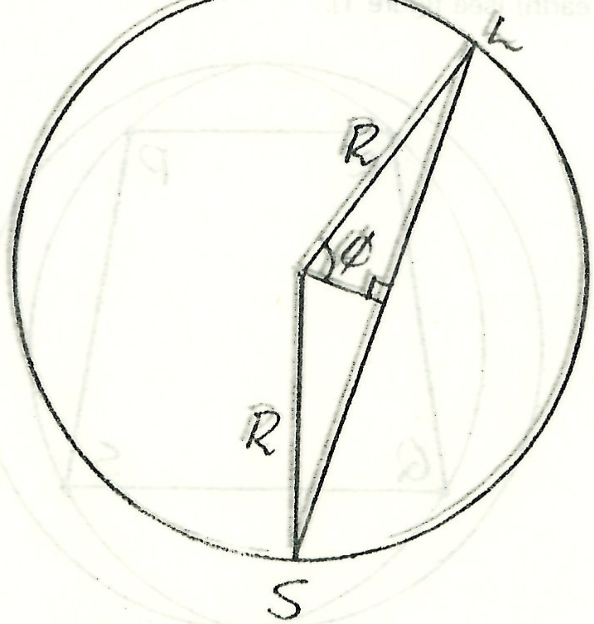


Figure 4

The angle ϕ is given by $\sin\phi = (LS/2)/R = LS/2R$,
or $\sin\phi = \sqrt{(\sin^2 43\frac{1}{2}^\circ + \cos 52^\circ \cos 35^\circ \sin^2 76^\circ)}$.
Then the great-circle distance between London and Sydney is given by

$$\text{great circle distance} = (2\phi)/(360^\circ) \times C,$$

where C is the circumference of the earth, and ϕ is measured in degrees.
From the above we find that

$$\begin{aligned} \sin\phi &= 0.96, \\ \phi &= 74^\circ, \\ \text{g.c. distance} &= 0.41C,\ C = 2\pi R = 40{,}000 \text{ km} \\ &= 0.41 \times 40000 \text{ km} \\ &= 16{,}400 \text{ km (to within 200 km)} \end{aligned}$$

☆

FINDING THE FIFTH ROOT OF A NUMBER

Jimmy Pike of Rydalmere has written to share with us a nice trick he has for finding the fifth root (if an integer) of a number up to eight digits long.

The fifth root of a number up to eight digits long has only one or two digits and is at most 39. From the last digit of the number, we get the last digit of the fifth root — they are equal. Then all we have to remember is that if the number has 6 digits, its fifth root has 2 digits, and that 16 is the smallest number whose fifth power has 7 digits, 26 is the smallest number whose fifth power has eight digits, and 36^5 starts with 6.

Exercise: What is the fifth root of 79235168?

WHAT'S THE TIME?

Jimmy also asks the following question, which you may like to research. Why do clockmakers use IIII rather than IV for 4 on Roman clockfaces?