PROBLEM SECTION

You are invited to submit solutions to one or more of these problems. Answers should bear your name, year, and school. Solutions will be published in the issue after next.

381. A square cake has frosting on its top and all four sides. Show how to cut it to serve nine people so that each one gets exactly the same amount of cake and exactly the same amount of frosting.

382. Prove or disprove: There are two numbers x,y such that $x + y = 1$, $x^2 + y^2 = 2$ and $x^3 + y^3 = 3$.

383. Let p(1), p(2), . . . p(n) be n points in the plane. Show that the shortest broken line connecting the points does not cross itself.

384. When the fire alarm went off, the 6 patrons in the restaurant all hurriedly seized a coat. Safely outside they discovered that no one had his own. The coat that Alf had belonged to the man who had seized Bert's. The owner of the coat grabbed by Colin held a coat which belonged to the man who was holding Dave's coat. If the man who had seized Ern's coat was not the owner of that grabbed by Fred, who borrowed Alf's coat? Whose coat did Alf seize? How do you know?

385. Let v be the number of vertices of a convex polyhedron, e the number of edges, and f the number of faces. Then Euler's formula is $v - e + f = 2$.

(i) Show that for any convex polyhedron $3f \leqslant 2e$ and $3v \leqslant 2e$ (Count the edges round each face, and at each vertex)

(ii) Prove or disprove: It is possible to cut a potato into a convex polyhedron having exactly seven edges.

386. Determine all polynomials $f(x) = ax^2 + bx + c$ such that

$$f(a) = a, \quad f(b) = b, \quad \text{and } f(c) = c.$$

387. Finitely many pennies are placed on a flat surface, no two overlapping. Prove or disprove: No matter how this is done, it is always possible to paint each penny with one of three colours so that no two pennies having the same colour touch each other.

388. **Let a list of integers a(1), a(2), . . ., a(n) be defined in succession by**
$a(n+1) = (a(n))^2 - a(n) + 1$ **and** $a(1) = 2$.

The first few are $a(1) = 2, a(2) = 3, a(3) = 7, a(4) = 43, a(5) = 1807, \ldots$
Show that the integers a(1), a(2), a(3), . . . are pairwise relatively prime (i.e. if a(k) and a(ℓ) are any two different members of the list, they have no common factor except 1).

389. For the same list of integers a(1), a(2), . . ., a(n), . . . in question 388 show that by taking N very large

$$|(1/a(1)) + (1/a(2)) + \ldots + (1/a(n)) - 1| < 1/10^{10}.$$

390. Let b(1), b(2), . . ., b(n) be any positive numbers
Prove that $(b(1) + b(2) + \ldots + b(n)((1/b(1))) + (1/b(2)) + \ldots + (1/b(n)) \geqslant n^2$

391. Each of three classes has n students. Each student knows altogether (n+1) students in the other two classes. Prove that it is possible to select one student from each class so that all three know one another. (Acquaintances are always mutual).

392. Let S consist of the set of all points (x,y) in the Cartesian plane such that x and y are both integers. The centre of gravity of the triangle with vertices (x(1),y(1)), (x(2),y(2)), (x(3),y(3)) is the point ((x(1) + x(2) + x(3))/3, (y(1) + y(2) + y(3))/3).

Prove that out of any 9 points in S, it is always possible to choose 3 with the property that the centre of gravity of the triangle formed by them is also a point in S.

Solutions to Problems from Vol. 13 No. 3

357. Chess-players from two schools competed. Each player played one game with every other player. There were 66 games among players from one school, and in all there were 136 games. How many players from each school entered the tournament?

Solution: This is the slightly condensed solution by Peter Crump (Sydney Grammar):

In general, for n players in a tournament the number of games played is $G = (n-1) + (n-2) + (n-3) + \ldots + 2 + 1 = \frac{1}{2}n(n-1)$ (by a formula for summing an arithmetical progression).

Altogether 136 games were played. Solving the quadratic equation $136 = \frac{1}{2}n(n-1)$ yields $n = 17$ (after discarding the other root, $n = -16$).

Similarly, since 66 games were played between players of one school the number of players in that school is a solution of

$$66 = \tfrac{1}{2}n(n-1), \text{ viz. } 12 \text{ players.}$$

Thus there are $(17-12) = 5$ players from the other school.