388. Let a list of integers a(1), a(2), . . ., a(n) be defined in succession by
$a(n+1) = (a(n))^2 - a(n) + 1$ and $a(1) = 2$.

The first few are a(1) = 2, a(2) = 3, a(3) = 7, a(4) = 43, a(5) = 1807, . . .
Show that the integers a(1), a(2), a(3), . . . are pairwise relatively prime (i.e. if a(k) and a(ℓ) are any two different members of the list, they have no common factor except 1).

389. For the same list of integers a(1), a(2), . . ., a(n), . . . in question 388 show that by taking N very large

$$|(1/a(1)) + (1/a(2)) + \ldots + (1/a(n)) - 1| < 1/10^{10}.$$

390. Let b(1), b(2), . . ., b(n) be any positive numbers
Prove that $(b(1) + b(2) + \ldots + b(n))((1/b(1)) + (1/b(2)) + \ldots + (1/b(n)) \geqslant n^2$

391. Each of three classes has n students. Each student knows altogether (n + 1) students in the other two classes. Prove that it is possible to select one student from each class so that all three know one another. (Acquaintances are always mutual).

392. Let S consist of the set of all points (x,y) in the Cartesian plane such that x and y are both integers. The centre of gravity of the triangle with vertices (x(1),y(1)), (x(2),y(2)), (x(3),y(3)) is the point ((x(1) + x(2) + x(3))/3, (y(1) + y(2) + y(3))/3).

Prove that out of any 9 points in S, it is always possible to choose 3 with the property that the centre of gravity of the triangle formed by them is also a point in S.

Solutions to Problems from Vol. 13 No. 3

357. Chess-players from two schools competed. Each player played one game with every other player. There were 66 games among players from one school, and in all there were 136 games. How many players from each school entered the tournament?

Solution: This is the slightly condensed solution by Peter Crump (Sydney Grammar):

In general, for n players in a tournament the number of games played is G = (n − 1) + (n − 2) + (n − 3) + . . . + 2 + 1 = ½n(n − 1) (by a formula for summing an arithmetical progression).

Altogether 136 games were played. Solving the quadratic equation 136 = ½n(n − 1) yields n = 17 (after discarding the other root, n = −16).

Similarly, since 66 games were played between players of one school the number of players in that school is a solution of

$$66 = \tfrac{1}{2}n(n-1), \text{ viz. } 12 \text{ players.}$$

Thus there are (17 − 12) = 5 players from the other school.

Essentially the same solution was supplied by J. Taylor (Woy Woy High School); P. Rider (St Leo's College, Wahroonga); S.S. Wadhwa (Ashfield Boys High School) and D. Dowe (Geelong Grammar School).

Comment: It is also just possible to interpret the problem to mean that in 66 games both players came from the same school (whether it was school A or B). There is a different solution possible with this interpretation, viz. 10 players from school A, 7 from school B, there being $\frac{1}{2}(10 \times 9) = 45$ games involving players from A only, and $\frac{1}{2}(7 \times 6)$ games involving players from B only, a total of $45 + 21 = 66$ such games.

358. What are the last two digits of $2^{2^{73}}$? Show your working.

Solution: P. Dowe (Geelong Grammar School) writes: (slightly expanded)

We want to find x such that $2^{2^{73}} \equiv x \pmod{100}$. Let us make a table of 2^n against r, the remainder when 2^n is divided by 100

n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
r =	2	4	8	16	32	64	28	56	12	24	48	96	92	84	68	36	72	44	88	76	52	4

We see that this repeats with a period of 20. [That is, the last two digits of 2^n are the same as those of 2^{20+n} if $n > 1$]. Let us find y such that $2^{73} \equiv y \pmod{20}$.

The last two digits of 2^{73} are the same as of 2^{53}, or 2^{33}, or 2^{13}, by the preceding remark, and from the table, they are 92. Hence the remainder y on dividing 2^{73} by 20 is 12.

Hence $2^{2^{73}} = 2^{20k+12}$ has the same last two digits as 2^{12} viz. 96.

Correct solutions also from S.S. Wadhwa and P. Crump.

One or two others incorrectly interpreted $2^{2^{73}}$ as $(2^2)^{73} = 4^{73}$ or 2^{146} instead of as the correct $2^{(2^{73})}$.

359. An infinitely long list is made of all the pairs of integers m,n for which $23m - 10n$ is exactly divisible by 17. Another list is made of all the pairs for which $7x + 11y$ is exactly divisible by 17. Prove that the two lists are exactly like.

Solution: D. Dowe (Geelong Grammar School) writes:

$23m - 10n \equiv 0 \bmod 17$ is equivalent to $6m + 7n \equiv 0 \bmod 17$ which is equivalent to $24m + 28n \equiv 0 \bmod 17$ which is $7m + 11n \equiv 0 \bmod 17$. As the other list gives x and y such that $7x + 11y \equiv 0 \bmod 17$, the two lists will be identical.

Also solved by J. Taylor (Woy Woy High School).

360. Suppose a(1), a(2), . . ., a(k) and b(1), b(2), . . ., b(k) are integers such that a(1) $\geqslant$ b(1) $\geqslant$ 1, a(2) $\geqslant$ b(2) $\geqslant$ 1, and so on. Let a = a(1) + a(2) + . . . a(k), and b = b(1) + b(2) + . . . + b(k).

(i) Prove that the product

$$[b(1)(a(1)-b(1)) + 1][b(2)(a(2)-b(2)) + 1] \times \ldots \times [b(k)(a(k)-b(k)) + 1]$$

is greater than or equal to a − b + 1.

(ii) Can you determine exactly under what conditions equality occurs?

Solution:

(i) If each A(n) $\geqslant$ 0

$$(1 + A(1))(1 + A(2))\ldots(1 + A(k)) = 1 + \sum_{n=1}^{k} A(n) + \sum_{1\leqslant i<j\leqslant k} A(i)A(j) + \sum_{i<j<\ell} A(i)A(j)A(\ell) + \ldots$$

$$\geqslant 1 + \sum_{n=1}^{k} A(n) \qquad (1)$$

since the omitted terms, involving products of two or more factors A(n), are all non-negative. Letting A(n) = b(n)(a(n) − b(n)), we have

$$[1 + b(1)(a(1)-b(1))][1 + b(2)(a(2)-b(2))]\ldots[1 + b(k)(a(k)-b(k))] \geqslant 1 + \sum_{n=1}^{k} b(n)(a(n)-b(n))$$

$$\geqslant 1 + \sum_{n=1}^{k} (a(n)-b(n)) \text{ (since } b(n) \geqslant 1\text{, and } a(n)-b(n) \geqslant 0) \qquad (2)$$

$\geqslant$ 1 + a − b as required.

(ii) Strict inequality would be obtained at (2) if in any term b(k) > 1 and a(k) − b(k) > 0. Further we already have strict inequality at (1) unless at most one of the A(n) exceeds 0 (e.g. If A(i) and A(j) are both > 0 the omitted term A(i)A(j) is positive). Hence the conditions for equality are that a(n) = b(n) for all save at most one value i of n, for which b(i) = 1.

e.g. a(n) = 2, 3, 6, 4, 5, for n = 1, 2, 3, 4, 5
and b(n) = 2, 3, 6, 1, 5

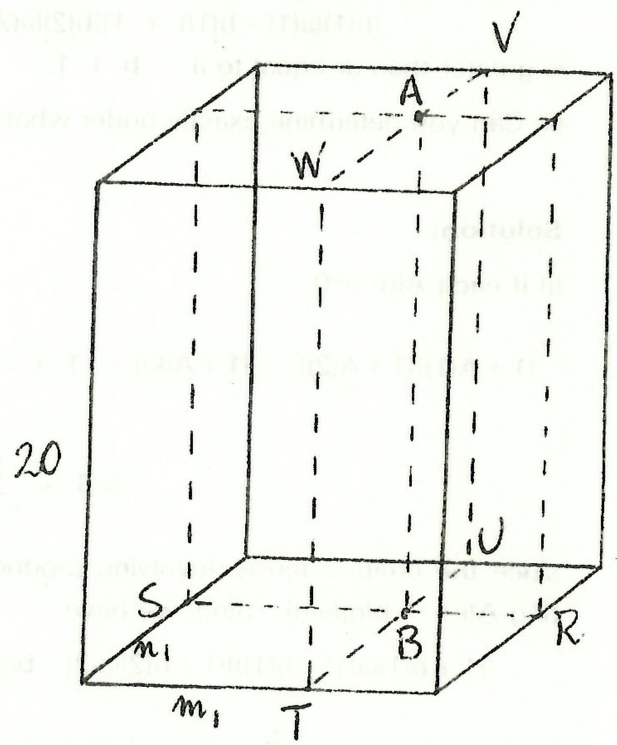
D. Dowe provided a correct proof of (1) and sufficient conditions for (ii) which however were not the most general possible.

J. Taylor's conditions for equality were likewise sufficient but not the most general.

361. A number of blocks, each 2 cm × 2 cm × 1 cm, have been fitted snugly together to make a solid 20 cm high. (The top dimensions of the solid are, say, m cm by n cm.) A straight line, parallel to the 20 cm sides pierces the solid from top to bottom. Prove that the straight line cannot pierce exactly one of the blocks.

Solution: One is meant to assume that the "snug" fitting not only means no cracks but also that the "solid" has plane faces i.e. it is a rectangular prism, since otherwise it is obviously very easy to construct a counter-example (say, 19 layers or stories consisting of 4 blocks "lying down" i.e. with the 1 cm side vertical, and the 20th layer consisting of a single block placed centrally).

Proceeding then on this asumption, suppose a vertical line AB **does** pass through only one block. Then it is clear first that AB is a whole number of centimetres from each face of the solid (since otherwise AB would pierce blocks at every level) and then that the block pierced, must be lying horizontally and pierced through the mid-point of its square face.



Imagine that the solid is now sawn along each of the planes TUVW and PQRS through AB parallel to the vertical faces of the solid (see figure). The block pierced by AB has been cut into equal quarters, and by our supposition, it is the only block so treated. Other blocks have possibly been cut exactly in halves by one cut or the other. But now we have an easy contradiction when we consider the volume of any one of the four corner solids which result from the cutting. For its volume is $20 \times m \times n$ cc's (where m and n are the integral number of cms in the lengths of BT and SB say); yet it is made up of a number of pieces of volume 4 cc's (whole blocks) some of volume 2 cc's (half blocks which have resulted from the sawing) and exactly one of volume 1 cc. Hence the volume of this corner would have to be an odd number of cc's which is not possible for $V = 20mn$. The desired result follows.

No solvers. I'm not surprised; this and 362 (ii) certainly gave me some trouble too.

362. (i) In the morning a working man leaves his cat in the house. The house has one door which has been left open. When the man returns in the evening the cat is outside. Prove that the cat crossed the threshold an odd number of times.

(ii) A triangle ABC is the union of a finite family, F, of triangles. If two different triangles in F intersect, they intersect in a vertex of both or an edge of both. Colour each of the vertices of the triangles in F red, blue or yellow. Colour A red, B blue, and C yellow. If a vertex V lies on AB, colour it red or yellow. Prove that the number of triangles in F which have one red, one blue and one yellow vertex is odd.

Solution: This obvious observation is intended to direct one's thinking at a certain stage in part (ii) of the question. It is hardly made more obvious by supplying a formal proof by mathematical induction or otherwise.

(ii) Figure 1 gives an example of the dissection of ABC into a family of triangles, F, satisfying the conditions F = {△'s BDY, DPY, DEP, EQP, ECQ, CXQ, XAQ, APX, and AYP}.

Triangles of F are not allowed to meet as in Figure 2 (i) or (ii).

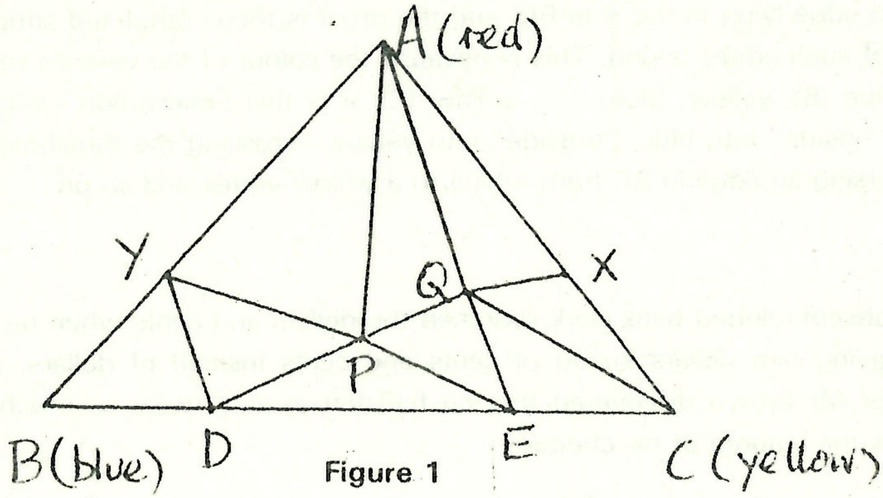


Figure 1

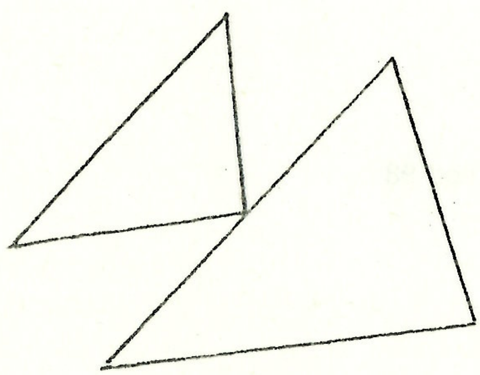


Figure 2(i)

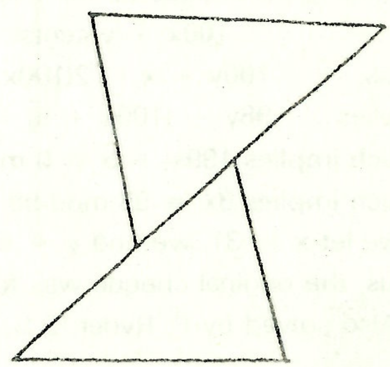


Figure 2(ii)

Imagine that the vertices of the triangles in fig. 1 have all been given a colour. If both vertices at the end of some edge of a triangle have the same colour we alter the figure by shortening this edge until it disappears, its two end points coalescing to form a single vertex still bearing that same colour.

For example if in figure 1 P and Q had been given the same colour, we move Q along PQ until the two points coincide. The triangles APQ and EPQ cease to exist and the triangle AXQ, CXQ and CBQ change size and shape.

This operation does not alter the number of triangles whose vertices are all of different colours, since the triangle or triangles which disappear certainly did not have that property. For the remainder of this answer it is assumed that this operation has been repeated as often as possible. We are unable to perform it only when all triangles remaining have a vertex of each of the three colours. You are reminded that the number of **such** triangles has not altered with any application of the operation. Each of the vertices A, B and C may have been fused with other vertices (for example, if E was assigned the yellow colour, it will at this stage have coalesced with C) but still been the same colour as previously. To finish we need to show that the total number of triangles remaining is odd. Consider the edges of triangles which have one vertex blue and one yellow. None occur on either the side AB or the side AC of the figure. If such an edge does not lie in the side BC, then it is the edge of intersection of two triangles, one on each side. This "pairs off" all triangles which do not have one of their edges lying in BC. There is just one triangle associated with each edge lying in the side BC, and the proof is then completed simply by observing that the number of such edges is odd. This is because the colour of the vertices remaining in BC alternates strictly blue (B), yellow, blue, . . . , yellow (C); it is this observation which is paralleled by 36 (i); translate "inside" into blue, "outside" into yellow, "crossing the threshold from inside to outside" into traversing an edge in BC from a blue to a yellow vertex and so on.

363. An absent-minded bank-clerk switched the dollars and cents when he cashed a cheque for Mr Brown, giving him dollars intead of cents and cents instead of dollars. After buying a five-cent newspaper Mr Brown discovered that he had left exactly twice as much as his original cheque. What was the amount of his cheque?

Solution: D. Dowe writes:

Let the initial cheque be for x dollars and y cents,

$= 100x + y$ cents.

Thus, $100y + x - 2(100x + y) - 5 = 0$

implies $98y - (100x + 5) = 0$

which implies $199x + 5 \equiv 0 \bmod 98$ and $3x + 5 \equiv 0 \bmod 98$

which implies $3x \equiv 93 \bmod 98$ and $x \equiv 31 \bmod 98$.

If we let $x = 31$, we find $y = 63$.

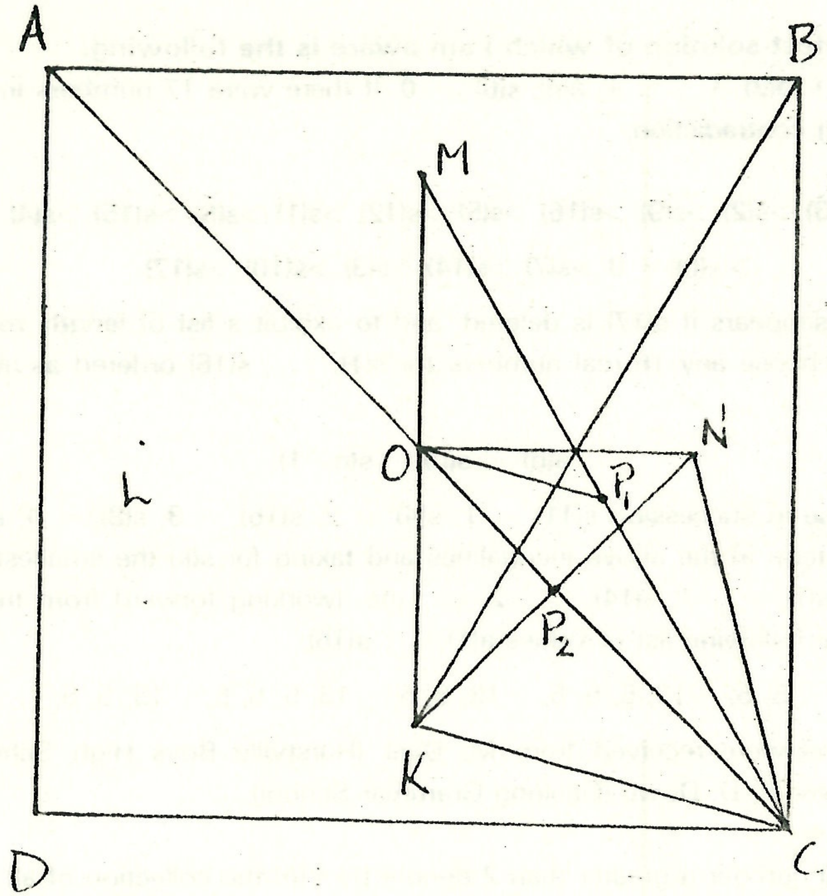
Thus, the original cheque was for \$31.63.

Also solved by P. Ryder, S.S. Wadhwa and P. Crump.

364. Four points K,L,M,N inside a square ABCD are such that ABK, BCL, CDM and DAN are all equilateral. Prove that the mid-points of the line segments KL, LM, MN, NK, AK, BK, BL, CL, CM, DM, DN, and AN are the vertices of a regular 12 sided polygon.

Solution: In the figure O is the centre of the square, and P(1), P(2) are the mid points of CM and KN respectively. We show that $OP(1) = OP(2)$ and that $\angle P(1)OP(2) = 30^\circ$. By the obvious symmetries of the figure about KM, LN, AC, and BD it then follows that all the twelve points are the same distance from O and that all the chords of the dodecagon subtend angles of 30° at O, whence it is a regular figure.

Since $BK = BC$ and $\angle KBC = 30^\circ$ we have $\angle BCK = \angle BKC = 75^\circ$ whence $\angle KCD = 15^\circ$. Similarly (or by symmetry about AC) $\angle BCN = 15^\circ$, leaving $\angle KCN = 90^\circ - 15^\circ - 15^\circ = 60^\circ$. It now follows that $\triangle CKN$ is equilateral, since there are many simple ways of seeing that $CK = CN$ (e.g. symmetry about AC, or congruence of $\triangle CKD$ and $\triangle CNB$). Now OP(1) is the join of mid points of two sides of $\triangle MKC$.



Therefore
$$OP(1) = \tfrac{1}{2} KC \qquad (1)$$
and OP(1) ∥ KC whence
$$\angle NOP(1) = \angle KCD = 15° \qquad (2)$$
Now P(2) is the centre of the semicircle on KN which because of the rightangle ∠KON passes through O.

Therefore
$$OP(2) = \tfrac{1}{2} KN, \qquad (3)$$
and
$$\angle NOP(2) = 45°. \qquad (4)$$
From (2) and (4)
$$\angle P(1)OP(2) = 45° - 15° = 30° \qquad (5)$$
and from (1) and (3) remembering that △CKN is equilateral we obtain
$$OP(1) = OP(2) \qquad (6)$$
As observed above, the desired result follows from (5) and (6).

365. a(1), a(2), a(3), . . . , a(N) is a list of real numbers such that the sum of every seven consecutive numbers in the list is negative but the sum of every 11 consecutive numbers is positive.

Find the maximum length of the list (i.e. the largest possible value of N) and give an example of such a list of maximum length.

Solution: The neatest solution of which I am aware is the following:

Set $s(n) = a(1) + a(2) + \dots + a(n)$, $s(0) = 0$. If there were 17 numbers in the list we would obtain the following contradiction.

$$s(17) > s(6) > s(13) > s(2) > s(9) > s(16) > s(5) > s(12) > s(1) > s(8) > s(15) > s(4) > s(11)$$
$$> s(0) = 0 > s(7) > s(14) > s(3) > s(10) > s(17)$$

The contradiction disappears if s(17) is deleted, and to exhibit a list of length 16 with the required properties, simply choose any 16 real numbers for s(1), . . ., s(16) ordered as above and then obtain each a(n) by

$$a(n) = s(n) - s(n-1).$$

For example, choose in succession $s(11) = 1$, $s(4) = 2$, $s(15) = 3$, $s(8) = 4$, etc. (working back from the $s(0) = 0$ term in the above inequalities and taking for s(k) the smallest available positive integer), and then $s(7) = -1$, $s(14) = -2$, . . . etc. (working forward from the zero term). This yields eventually the following list of values a(1), . . . a(16)

$$5,\ 5,\ -13,\ 5,\ 5,\ 5,\ -13,\ 5,\ 5,\ -13,\ 5,\ 5,\ 5,\ -13,\ 5,\ 5.$$

Excellent solutions were received from M. Dyer (Hurstville Boys High School), M. Reynolds (Marist Bros, Pagewood), D. Dowe (Geelong Grammar School).

366. For any natural number n greater than 2 denote by V(n) the collection of all numbers expressible in the form $1 + kn$ where k is a positive integer. A number x in V(n) will be called indecomposable if there do not exist elements, y, z in V(n) such that $x = yz$ (e.g. 25 is indecomposable in V(3)). Show that V(n) contains numbers which can be factorised in two different ways into indecomposable factors (i.e. it is possible to have $uv = wx$ in V(n) where all of u, v, w and x are indecomposable but $u \neq w$, $u \neq x$).

Solution: Matthew Dyer (Hurstville Boys High School) has sent in the following beautiful solution:

Every decomposable element of V(n) is of the form $(k_1 n + 1)(k_2 n + 1)$. The five smallest decomposable elements of V(n) are (in order of size) $R_1(n) = (n+1)^2$, $R_2(n) = (n+1)(2n+1)$, $R_3(n) = (n+1)(3n+1)$, $R_4(n) = (2n+1)^2$, $R_5(n) = (n+1)(4n+1)$. Any other decomposable element would be larger since in its expansion as $An^2 + Bn + C$, we would clearly have $A > 4$, $B \geqslant 5$, $C = 1$, and so the number would be greater than $R_5(n)$. In particular, any decomposable number smaller than $R_4(n)$ is either $R_1(n)$, $R_2(n)$ or $R_3(n)$.

Now consider the numbers (all unequal)

$X(n) = (n-1)^2$, $Y(n) = (2n-1)^2$, $Z(n) = (n-1)(2n-1)$.

Since $n > 2$, all three are clearly members of V(n). X(n) is indecomposable in V(n), since $X(n) < R_1(n)$. Now $Y(n) < R_4(n)$, so if Y(n) is decomposable, then $Y(n) = R_3(n)$, $Y(n) = R_2(n)$ or $Y(n) = R_1(n)$. Solving the resulting quadratic equations for integral $n > 2$, we find Y(n) is decomposable in V(n) iff $n = 8$. Similarly, Z(n) is decomposable in V(n) iff $n = 5$.

Thus if $n \neq 5, 8$, then $X(n)$, $Y(n)$, $Z(n)$ are three unequal indecomposable members of $V(n)$, and $X(n) = Z(n)$.

If $n = 5$, $16 \times 361 = 76 \times 76$ where $16 = 4^2$, $361 = 19^2$ and $76 = 4 \times 19$ are clearly indecomposable members of $V(n)$.

If $n = 8$, $25 \times 169 = 65 \times 65$ where $25 = 5^2$, $169 = 13^2$ and $65 = 5 \times 13$ are indecomposable in $V(n)$.

Thus $V(n)$ always contains elements which can be factorised two different ways into products of indecomposable factors.

367. Let a and b be positive integers such that, when $a^2 + b^2$ is divided by $a + b$ the quotient q and the remainder r satisfy the equation $q^2 + r = 1977$. Find all possible values of a and b.

Solution: Matthew Dyer (Hurstville Boys High School) writes:

We are given

$$a^2 + b^2 = (a + b)q + r \quad (0 \leqslant r < a + b) \qquad (1)$$

$$q^2 + r = 1977 \qquad (2)$$

Substituting for r from (2) into (1), we obtain $q^2 - (a + b)q + (a^2 + b^2 - 1977) = 0$.
Since q is real, this quadratic equation in q must have real roots.
Therefore $(a + b)^2 - 4(a^2 + b^2) + 4 \times 1977 \geqslant 0$, which gives $3a^2 - 2ab + 3b^2 \leqslant 4 \times 1977$.
Geometrically, this is the region interior to and on the ellipse $3a^2 - 2ab + 3b^2 = 4 \times 1977$ in the (a,b)-plane (with rectangular coordinates). The maximum value of $a + b$ in this region will clearly be on the ellipse. By using Lagrange multipliers, or otherwise, this maximum is found to be $2\sqrt{1977} < 89$.

Hence $\quad a + b < 89. \qquad (3)$

Now if $\quad q \geqslant 45$, $q^2 > 1977$, contradicting (2) since by (1), $r \geqslant 0$.
Hence $\quad q \leqslant 44$. But if $q \leqslant 43$, $r \geqslant 1977 - 43^2 = 128$, whence by (1), $a + b > r \geqslant 128$, contradicting (3).

Therefore $q = 44$, and by (2), $r = 41$.

(1) now becomes $a^2 + b^2 = 44(a + b) + 41$ or $a^2 - 44a + (b^2 - 44b - 41) = 0. \qquad (4)$
Since a is integral, the discriminant $D = 44^2 - 4(b^2 - 44b - 41) = 4(525 + 44b - b^2)$ must be a perfect square. By trial, D is a perfect square only for $b = 7$, 37 and 50. The corresponding possibilities for a are $a = 50$, 50 and 22 ± 15. Thus the only possible values of a and b satisfying the condition are ($a = 50$, $b = 37$), ($a = 50$, $b = 7$), ($a = 7$, $b = 50$) and ($a = 37$, $b = 50$). It is found that these pairs do in fact satisfy the given conditions.

A correct answer was received also from S.S. Wadhwa.

368. The infinite sequence

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

consists of positive integers, (i.e. f(n) is always a positive integer) and satisfies the inequality $f(n+1) > f(f(n))$ for every positive integer n. Prove that $f(n) = n$ for every n.

Solution: Matthew Dyer (Hurstvile Boys High School) writes:

Suppose that there exists no integer j such that $f(j) = 1$. Then $f(k) \geqslant 2$ for all k so
$f(2) > f[f(1)] > f[f(k_1)]$ where $k_1 = f(1) - 1 \geqslant 1$ since $f(1) \geqslant 2$
$f[f(k_1)] > f[(k_2)]$ where $k_2 = f(k_1) - 1 \geqslant 1$ since $f(k_1) \geqslant 2$
$f[f(k_2)] > f[f(k_3)]$ where $k_3 = f(k_2) - 1 \geqslant 1$ since $f(k_2) \geqslant 2$.
etc. at each stage, $k_{n+1} = f(k_n) - 1 \geqslant 1$, since $f(k) \geqslant 2$, for all k implies $f(k_n) \geqslant 2$ whatever the value of k_n. Hence $f(k_{n+1})$ and so $f[f(k_{n+1})]$ is defined and is a positive integer so the process may be continued indefinitely yielding an infinite strictly decreasing sequence of positive integers, which is impossible since there are only a finite number of positive integers less than $f(2)$.

Hence $f(j) = 1$ for some integer j. If $j \neq 1$, then $j \geqslant 2$ and so $f(j-1)$ is defined and is a positive integer, and hence so also is $f[f(j-1)]$. But $f(j) = 1 > f[f(j-1)]$ which is impossible since there is no positive integer less than 1. Hence $j = 1$. We have shown that there is an integer j such that $f(j) = 1$, and that if $f(j) = 1$, then $j = 1$. i.e. we have proved the proposition $D(1) : f(j) = 1$ iff $j = 1$. We use $P(1)$ as a basis for an inductive proof of the required result.

Assume
$P(k) : f(j) = 1$ iff $j = 1$, $f(j) = 2$ iff $j = 2$, $\ldots$ $f(j) = k$ iff $j = k$ is true, where k is a positive integer.

The equation in j $f(j) = k+1$ implies $j = k+1$; for $k+1 = f(j) > f[f(j-1)]$ and so $f[f(j-1)] = 1$, $2 \ldots$ or k. By the inductive hypothesis, we would then have $f(j-1) = 1, 2, \ldots$ or k, $j-1 = 1, 2 \ldots$ or k, and $j = 2, 3 \ldots$ or $k+1$. Of these, the only possibility satisfying the original equation is $j = k+1$, for if $j \leqslant k$, $f(j) = j < k+1$. Thus $f(j) = k+1$ only if $j = k+1$.

Now suppose $f(k+1) \neq k+1$. Then $f(p) \neq k+1$ holds for all p. From the inductive hypothesis, $f(p) \neq 1, 2 \ldots k$ for $p \geqslant k+1$. Hence, if $p \geqslant k+1$, $f(p) \geqslant k+2$. Then $f(k+2) > f[f(k+1)] > f[f(k_1)]$ where $k_1 = f(k+1) - 1 \geqslant k+1$.
$f[f(k_1)] > f[f(k_2)]$ where $k_2 = f(k_1) - 1 \geqslant k+1$
$f[f(k_2)] > f[f(k_3)]$ where $k_3 = f(k_2) - 1 \geqslant k+1$
etc. Again we would obtain an infinite strictly decreasing sequence of positive integers, which is impossible. Hence the assumption that $f(k+1) \neq k+1$ leads to an impossibility.
Therefore $f(j) = k+1$ if $j = k+1$. But we have already shown that $f(j) = k+1$ only if $j = k+1$. Thus the proposition $P(k+1) : f(j) = 1$ iff $j = 1$, $f(j) = 2$ iff $j = 2, \ldots, f(j) = k+1$ iff $j = k+1$ is true. i.e $P(k)$ implies $P(k+1)$. But we have already shown $P(1)$ is true. Thus, by mathematical induction $P(k)$ is true for all positive integral k.

In particular, the proposition $f(n) = n$ is true. This is the required result.