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THE GEOMETRIC NATURE OF π

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π is an important universal constant and appears in many fields of mathematics and theoretical physics. It is an irrational number which is not a root of any algebraic equation; hence it is known as a transcendental number. It is usually determined, as accurately as we require, with the help of sophisticated mathematical tools such as Taylor's Series or Fourier Series, and the use of computers to do the arithmetic has now enabled us to determine the value of π to more than a million decimal places — a fairly arid achievement.

Yet π is, of course, associated with a very simple and basic ratio — that of the circumference to the diameter of a circle, and in this sense its more-or-less approximate value has been known for millenia. Probably the most advanced method of its geometric determination still stands to the credit of Archimedes who lived two thousand years ago. He attempted to find the perimeter of a circle, and hence the value of π, by considering the circle as the limit of a regular n-sided polygon as n becomes indefinitely large. It is interesting and instructive to follow in Archimedes' footsteps in looking at this problem.

Consider first a square with four sides of length 2r; r can be considered as the 'radius' of the square since it is the perpendicular distance of each side from the centre (see figure 1). Hence for this square the perimeter, P(4), and the area, A(4), relating to this regular 4-sided polygon, are given by

$$P(4) = 8r = 2\pi(4)r,$$

and $$A(4) = 4r^2 = \pi(4)r^2,$$

where $$\pi(4) = 4.$$

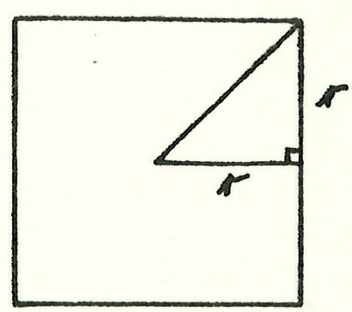

Figure 1

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Consider next a regular hexagon ($n = 6$) with r defined in the same way as for the square (see figure 2). The length, s, of each side is related to r by

$$s/2 = r \tan 30^\circ = r/\sqrt{3}$$

so that $P(6) = 12\, r/\sqrt{3} = 2\pi(6)r$,

and $A(6) = 6r^2/\sqrt{3} = \pi(6)r^2$,

where $\pi(6) = 2\sqrt{3} \cong 3.464$.

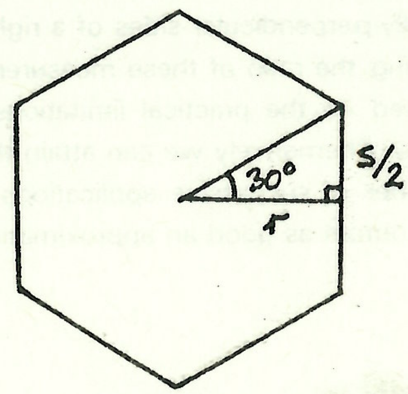

Figure 2

For a regular octagon ($n = 8$), we have similarly $s/2 = r \tan 22\frac{1}{2}^\circ$. We can easily obtain the exact value of $\tan 22\frac{1}{2}^\circ$ using the trigonometric formula

$$\tan\theta = \sqrt{(\cot^2 2\theta + 1)} - \cot 2\theta. \qquad \text{(A)}$$

For $\theta = 22\frac{1}{2}^\circ$, $\cot 2\theta = 1$, so we find that

$$\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1,$$

hence $P(8) = 8s = 16r(\sqrt{2}-1) = 2\pi(8)r$,

and $A(8) = 8r^2(\sqrt{2}-1) = \pi(8)r^2$,

where $\pi(8) = 8(\sqrt{2}-1) \cong 3.314$.

Let us take one more case, say $n = 12$, before we start generalising.

Here $s/2 = r \tan 15^\circ$,

where $\tan 15^\circ = \sqrt{(\cot^2 30^\circ + 1)} - \cot 30^\circ = 2 - \sqrt{3}$

hence $P(12) = 12s = 24r(2-\sqrt{3}) = 2\pi(12)r$

and $A(12) = 6sr = 12r^2(2-\sqrt{3}) = \pi(12)r^2$

where $\pi(12) = 12(2-\sqrt{3}) \cong 3.215$.

It is seen that for an n-sided regular polygon

$$P(n) = ns = 2nr\tan(180^\circ/n) = 2\pi(n)r$$

and $A(n) = \frac{1}{2}nsr = nr^2 \tan(180^\circ/n) = \pi(n)r^2$

where $\pi(n) = n \tan(180^\circ/n)$,

so that $\pi = \lim_{n\to\infty} n \tan(180^\circ/n)$

(Note that we have scrupulously avoided the use of radian measure in describing the angles involved. The above geometric approach is quite independent of the radian measure concept.)

Our approach provides a simple explanation of the intimate relation between the formulae $P = 2\pi r$ and $A = \pi r^2$.

$$P(n) = ns, \text{ while } A(n) = \tfrac{1}{2}nsr,$$

so $A(n) = \frac{1}{2}rP(n)$.

It is seen that this geometric approach provides a simple arithmetic procedure for calculating π as accurately as required. It is merely a matter of taking n sufficiently large and determining the value of n tan (180°/n). In particular this could be done by direct measure of the lengths of the two mutually-perpendicular sides of a right-angled triangle having one angle equal to (180°/n), and then obtaining the ratio of these measurements; however in practice the efficacy of such a procedure is restricted by the practical limitations in the precision of constructing right-angles and measuring lengths. Alternatively we can attain the exact value (in terms of surds) of tan (180°/n) for large n by a number of successive applications of the formula (A), and in this way, with the help of a computer, obtain as good an approximation of π as by any other method. You might like to try it.

CORRIGENDA

Unfortunately, a number of annoying misprints appeared in Vol. 14 No. 2. Here are the corrections:-

p.8 Expectation of life $= \frac{1}{2} + (\ell_1 + \ell_2 + \ldots)/\ell_0$.

Expectation of life of those alive at age x $= \frac{1}{2} + (\ell_{x+1} + \ell_{x+2} + \ldots)/\ell_x$.

p.12 $f(x) = 2x + 1$

p.13 Problem 2 $t = -\frac{1}{2}$

Problem 3 $x + y - 2 + \lambda(2x + y + 1) = 0$

p.14 Problem 11 $= (1/a)\ln(ax + b) + C$

Problem 12 $x^2 + 2x + 1 > 1, \quad x(x + 2) > 0$

I again invite you to tell us the **mathematical** errors in the "solutions" to the problems in Dr Karpilovsky's article.