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# APPROXIMATE ANGLE-TRISECTION

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## §1. Introduction

Angle-Trisection has a history of its own. It has aroused much interest in the minds of students and teachers alike. It has been proved that (see E.E. Moise, Elementary Geometry from an Advanced Standpoint (Addison-Wesley 1963)) to exactly trisect any unknown angle by a Euclidean procedure (ruler-and-compass-construction) is impossible. However, many fertile brains have tried to solve this famous problem from antiquity and the curiosity has not ceased even now. As a result, there are available several methods which can be broadly classified into three categories, viz.,

(i) Euclidean methods yielding approximate results,

(ii) methods based on certain curves, for example the conchoid, limaçon, trisectrix, and

(iii) methods involving some processes (for example revolving a scale, examining the collinearity of three points) which may be called non-Euclidean.

Here we will discuss a procedure of the first kind based on the geometrical solution of a cubic equation.

## §2. The Cubic Equation

If c be the length of a chord AB which subtends an arbitrary angle $\theta = 3\alpha$ ($\leqslant 180°$) at the centre O of a circle of radius r and x be the length of the chord CD subtending the angle $\theta/3 = \alpha$ ($\leqslant 60°$) at the centre of the same circle (figure 1), then a determination of x in terms of r and c will enable us to effect the trisection of the angle $\theta$; because the problem lies in trisecting the arc AB. Let the two trisectors OC and OD of the angle $\theta$ cut AB at E and F respectively.

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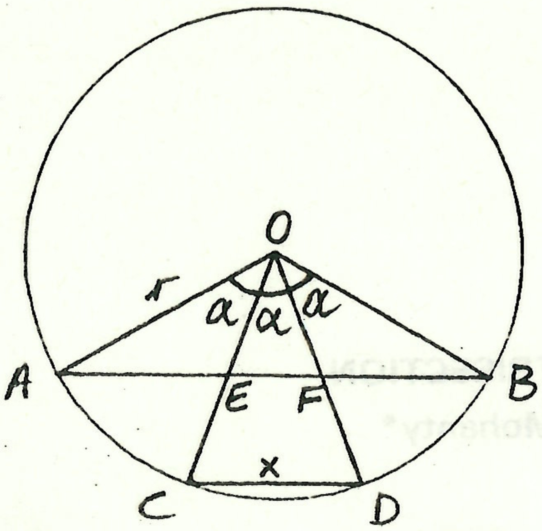


Figure 1

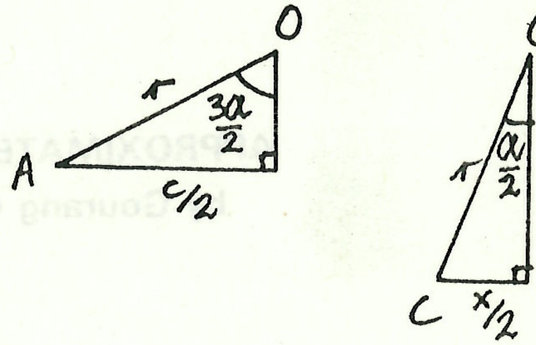


Figure 2

It is easily observed that (see figure 2)

$$c/2r = \sin(3\alpha/2) = 3\sin(\alpha/2) - 4\sin^3(\alpha/2)$$

and $\quad x/2r = \sin(\alpha/2).$

From these eliminating $\alpha$ we have

$$x^3 - 3xr^2 + cr^2 = 0 \qquad (1)$$

This is the desired equation and we require to solve it geometrically.

For convenience we substitute

$$C = c/r \text{ and } X = x/r \qquad (2)$$

The equation (1) can be written in the simpler form

$$X^3 = 3X - C \qquad (3)$$

**§3. Solving the Cubic**

As we know, the root of equation (3) is the abscissa of the point of intersection of the two graphs

$$\text{(i)} \quad Y = 3X - C \qquad \text{and (ii)} \quad Y = X^3$$

Obviously (i) is a straight line of slope 3 passing through the point $(C/3,0)$ and can be drawn easily by ruler and compass for every value of C. However, this is not the case with the cubic curve (ii) and hence it presents a little difficulty which is to be surmounted.

But it is a fact that if $(X_1, Y_1)$ and $(X_2, Y_2)$ are two points on the graph of $Y = f(X)$ and if $X_1$ is sufficiently near to $X_2$ then the graph within the interval is very near the straight line connecting these two points. This is more valid an assumption in case of the 'steep' cubic curve we are concerned with. Therefore, within a 'small' interval $a < X_0 < b$ (where $X_0$ is the exact root of equation (3)), our cubic curve can be approximated by the straight line joining the point $A(a,a^3)$ and $B(b,b^3)$ which can be very easily found by a Euclidean procedure (see Moise) for all values of a and b. The abscissa $X'_0$ of the point of intersection of the straight line (i) and the straight line AB can now be determined and this furnishes an approximation $X'_0$ for $X_0$ (see figure 3).

Hence all that is required is to choose the interval $|a-b|$ as small as possible. This is achieved as follows.

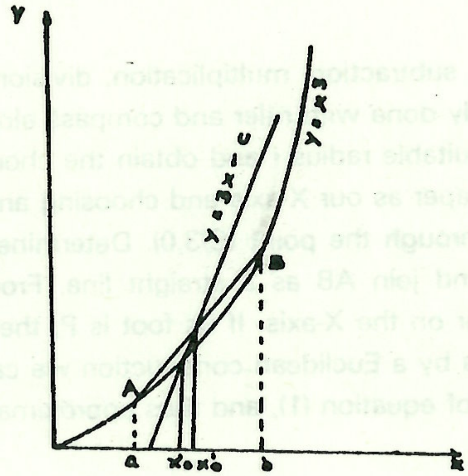


**Figure 3**

## §4. The Interval

Now from figure 1 it can be easily seen that AEDC is a rhombus and DCEF is an isosceles trapezium, and hence we obtain

$$AE = CD = BF \text{ and } EF \parallel CD.$$

Also CE and DF meet at O.

Therefore, CD > EF. That is, AE = BF > EF.

Hence EF < AB/3, that is, AE = BF > AB/3.

Again E and F can not coincide; for then the angle $\theta$ will be bisected.

Hence AE = BF < AB/2. Obviously then

$$c/3 < x < c/2$$

(It may be noted here that for $\theta = 0°$, E and F will coincide with C and D respectively and then $x = c/3$. Similarly for $\theta = 180°$, E and F will coincide and then $x = c/2$. These cases are trivial and can be excluded from our consideration.)

But the interval specified as above is rather large and we can further contract it. To do so, we observe that x increases monotonically from c/3 to c/2 as $\theta$ varies steadily from 0° to 180°. Considering three equally spaced values of x within this interval, $x_1 = 9c/24$, $x_2 = 10c/24$, $x_3 = 11c/24$, there must correspond three values of $\theta$ such that for $\theta = \theta_1$, $x = x_1$; for $\theta = \theta_2$, $x = x_2$; and for $\theta = \theta_3$, $x = x_3$. Let us evaluate $\theta_1$, $\theta_2$ and $\theta_3$.

Putting $x_1 = 9c/24$ for x in equation (1) we obtain after solution,

$$c/r \cong 1.532 = 2 \sin(\theta_1/2) \text{ which implies } \theta_1 \cong 100°.$$

Similarly we can show that $\theta_2 \cong 136.5°$ and $\theta_3 \cong 161°$. Note that if we can trisect an acute angle $\theta$, we can also trisect an obtuse angle $90^\circ + \theta$; it is enough, therefore, to consider, without loss of generality, only the range $c/3 < x < 3c/8$ which covers all acute angles and thus we obtain our required interval,

$$C/3 < X_0 < 3C/8, \quad \text{that is,} \quad a = C/3 \text{ and } b = 3C/8 \tag{4}$$

**§5. Constructions**

It is well known that addition, subtraction, multiplication, division, extraction of integral powers and square-roots etc. can be easily done with ruler and compass alone (see Moise). Thus given the angle $\theta$, describe an arc with a suitable radius r and obtain the chord c; then obtain $C = c/r$. Taking any straight line on a plane paper as our X-axis and choosing any point O upon it as the origin, draw a straight line of slope 3 through the point (C/3,0). Determine the points $A(a,a^3)$ and $B(b,b^3)$ with $a = C/3$ and $b = 3C/8$ and join AB as a straight line. From the intersection of the two straight lines drop a perpendicular on the X-axis. If its foot is P, then $X'_0 = \overline{OP}$ is an approximate solution of equation (3). From this by a Euclidean construction we can determine $x'_0 = rX'_0$ as an approximation to $x_0$, the solution of equation (1), and thus approximately trisect the angle.

**§6. Error Estimation**

Let us examine how much error has crept in through our approximation in replacing the cubic curve by the straight line. The equation of the straight line AB joining the points $(a,a^3)$ and $(b,b^3)$ is

$$Y = X(a^2 + ab + b^2) - ab(a + b).$$

The abscissa of its intersection with $Y = 3X - C$ is given by

$$X'_0 = (C - ab(a + b))/(3 - (a^2 + ab + b^2))$$

Putting $a = C/3$ and $b = 3C/8$ and noting that $C = 2\sin(\theta/2)$, we obtain

$$X'_0 = ⅓ \times 2\sin(\theta/2) \times (432 - 153\sin^2(\theta/2))/(432 - 217\sin^2(\theta/2)) \tag{5}$$

The error

$$\delta = X'_0 - X_0 < X'_0 - C/3 \quad \text{since } X_0 > C/3.$$

That is,

$$\delta < (128/3) \times \sin(\theta/2)/(432 - 217\sin^2(\theta/2))$$

and the relative error

$$\delta/X_0 \cong \delta/X'_0 < 64/(432 - 217\sin^2(\theta/2)) \tag{6}$$

Equation (6) clearly indicates that the upper limit of error increases as the angle gets larger. Hence it is sufficient if we consider the case of maximum error which corresponds to $\theta = 90^\circ$. For this case equation (5) yields $X'_0 \cong 0.518113$.
But $90^\circ$ is a trisectable angle for which

$$X_0 = 2\sin 15^\circ = (\sqrt{3} - 1)/\sqrt{2} \cong 0.517714.$$

Hence $\delta \cong 0.000399$ and $\delta/X_0 \cong 0.000770$.
Since $90^\circ$ corresponds to maximum error we may write safely that

$$\delta < 0.0004 \quad \text{and} \quad \delta/X_0 < 0.0008.$$

Thus my method enables one to trisect an angle with an error of less than 0.08%!