LETTERS TO THE EDITOR

Sir,

Recently I came across the following problem:

"I have a circular field radius 100 metres, and a goat. If the goat is tethered from a rope at the periphery of the field so that it can graze on exactly one-half of the field, how long is the rope?"

By finding an equation for the area of field grazed in terms of the length of the rope, I had hoped to be able to solve the problem. The equation I developed seems impossible to solve except by successively closer approximations.

Ross Baldick,
Year 11, Chatswood High.

Editor's Comments: Let the radius of the field be r, the length of the rope be R, the centre of the field be O and the point where the goat is tethered be P (see figure 1).

In $\triangle OAP$, $OA = OP = r$,
so $\angle OAP = \angle OPA = \theta$,
so $\angle APB = 2\theta$,
and $\angle AOP = \pi - 2\theta$,
so $\angle AOB = 2\pi - 4\theta$.

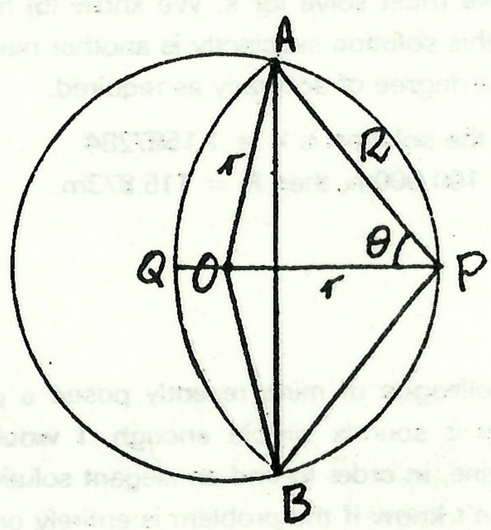

Figure 1

The area of the segment AQB of the circle centred at P is given by

$$\tfrac{1}{2}R^2(2\theta - \sin 2\theta),$$

(try to prove this!)

while the area of the segment APB of the circle centred at O is given by

$$\tfrac{1}{2}r^2((2\pi - 4\theta) - \sin(2\pi - 4\theta)).$$

We are told that

$$\tfrac{1}{2}R^2(2\theta - \sin 2\theta) + \tfrac{1}{2}r^2((2\pi - 4\theta) - \sin(2\pi - 4\theta)) = \tfrac{1}{2}\pi r^2, \quad (1)$$

or,

$$(R/r)^2(2\theta - \sin 2\theta) + ((2\pi - 4\theta) - \sin(2\pi - 4\theta)) = \pi. \quad (2)$$

If we write $k = R/r$, and use the fact that $\sin(2\pi - 4\theta) = -\sin 4\theta$, we obtain

$$k^2(2\theta - \sin 2\theta) + ((2\pi - 4\theta) + \sin 4\theta) = \pi, \quad (3)$$

or,

$$(4 - 2k^2)\theta + k^2 \sin 2\theta - \sin 4\theta = \pi. \quad (4)$$

Now, $\cos\theta = (R^2 + r^2 - r^2)/2 \times R \times r$ (see figure 2)

$= R^2/2Rr$

$= k/2,$

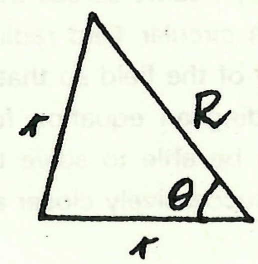

Figure 2

from which it follows that

$\sin 2\theta = 2\sin\theta\cos\theta = \tfrac{1}{2}k\sqrt{(4-k^2)},$

and

$\sin 4\theta = 2\sin 2\theta \cos 2\theta = \tfrac{1}{2}k(k^2-2)\sqrt{(4-k^2)}.$

So equation (4) becomes

$$(4-2k^2)\cos^{-1}(\tfrac{1}{2}k) + k\sqrt{(4-k^2)} = \pi \quad (5)$$

This we must solve for k. We know (5) has just one solution for k positive, but whether one can state this solution succinctly is another matter. I cannot. But we can approximate the solution to as great a degree of accuracy as required.

Thus, the solution is $k \cong 1.1587284$.
If $r = 100.000$m, then $R = 115.873$m.

Sir,

A colleague of mine recently posed a problem to me, which I have found very difficult, even though it sounds simple enough. I would like to submit the problem for publication in your magazine, in order to find an elegant solution for it.

I don't know if the problem is entirely original, or even if similar problems may have appeared in issues I have missed.

The problem concerns two circular intersecting racing car circuits, whose radii are 4 and 2 units respectively, and whose centres are 5 units apart.

Two racing cars start from points diametrically opposite A and B (see figure 3).

The car starting from point A, travels anticlockwise with speed V_1 and the car which starts from point B travels clockwise with speed V_2.

The speeds of the cars are such that

$$V_2 : V_1 = 3 : 1$$

The question is do the cars ever collide — if they do, how many revolutions have been traversed?

A. Thompson,
Smiths Hill Girls High.

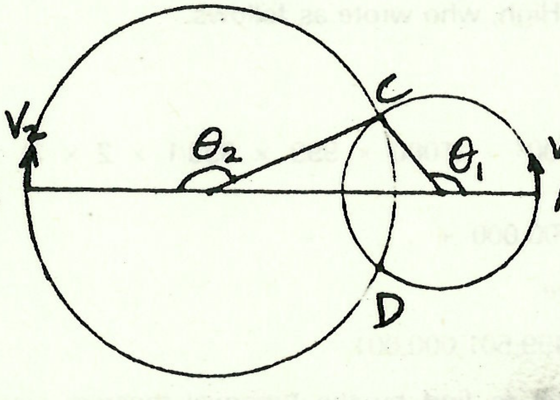

Figure 3

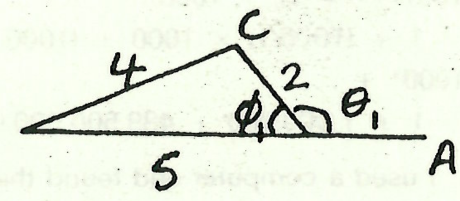

Figure 4

Editor's Comments: No, the cars do not collide.

Suppose they do collide at C (see figure 1). Then car (A) has travelled through n revolutions (n an integer) plus the angle θ_1, and so has travelled a distance of $(n \cdot 2\pi + \theta_1)r_1$ and has taken $(n \cdot 2\pi + \theta_1)r_1/V_1$ units of time to do so. We can write

$$(n \cdot 2\pi + \theta_1)r_1/V_1 = (m \cdot 2\pi + \theta_2)r_2/V_2,$$

or,

$$(2n\pi + \theta_1)/(2m\pi + \theta_2) = V_1 r_2/V_2 r_1 = 2/3,$$

or, after some simplification,

$$3\theta_1 - 2\theta_2 = (2m - 3n) \cdot 2\pi \qquad (1)$$

Similarly, if the cars collide at D we can deduce that

$$3\theta_1 - 2\theta_2 = (2p - 3q)2\pi, \qquad (2)$$

where p and q are integers.

Now, in both (1) and (2), the right hand side is a multiple of 2π.

However, from figure 4, we see that

$$\cos\phi_1 = (2^2 + 5^2 - 4^2)/2 \times 2 \times 5 = 13/20,$$

so $\phi_1 < \pi/3$, whence $\theta_1 > 2\pi/3$.

Also $\theta_2 < \pi$,

so $3\theta_1 - 2\theta_2 > 0$,

and trivially $\theta_1 < \pi$, $\theta_2 > \pi/2$,

so $3\theta_1 - 2\theta_2 < 2\pi$.

We see that $3\theta_1 - 2\theta_2$ is **not** a multiple of 2π, so the cars do not collide.

In a more general situation, as long as the quantity $V_1 r_2/V_2 r_1$ is rational, it should be fairly easy to tell whether or not the cars collide. However, if $V_1 r_2/V_2 r_1$ is irrational ...