TRY THIS FOR SIZE

In Vol. 14 No.'s 1 and 2, the problem of finding the digits of 1001^{1000} was posed. A good attempt was made by Ross Baldick, year 11, Chatswood High, who wrote as follows.

Using the binomial theorem,

$1001^{1000} = (1 + 1000)^{1000}$

$= 1 + (1000/1) \times 1000 + (1000 \times 999/1 \times 2) \times 1000^2 + (1000 \times 999 \times 998/1 \times 2 \times 3) \times 1000^3 + \ldots$

$= 1 + 1{,}000{,}000 + 499{,}500{,}000{,}000 + 166{,}167{,}000{,}000{,}000 + \ldots$

I used a computer and found that the last 48 digits are

$\ldots 324{,}359{,}331{,}903{,}843{,}040{,}069{,}467{,}324{,}167{,}490{,}917{,}499{,}501{,}000{,}001.$

The first (and most significant) digits are more difficult to find by the binomial theorem since "carry" from less significant digits presents a problem. However, we can obtain

$1001^{1000} = 1000^{1000} \times (1.001)^{1000}$

$= 10^{3000} \times (1 + 1000 \times 0.001 + (1000 \times 999/1 \times 2) \times (0.001)^2 +$
$+ (1000 \times 999 \times 998/1 \times 2 \times 3) \times (0.001)^3 + \ldots)$

$= 10^{3000} \times (1 + 1 + 0.4995 + 0.166167 + 0.04141712475 + 0.0082502912502 +$
$+ 0.0013681732989115 + \ldots)$

$= 10^{3000} \times 2.71 \ldots.$

Editor's Comments: One can do somewhat better with the leading digits if one knows some analysis. There is a number called "e" which you will learn about. It can be defined in the following way: The graph of $y = e^x$ has slope precisely 1 at the point (0,1).

The following facts can be proved:

(1) For every x, $e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + \ldots$

In particular, $e = 1 + 1/1! + 1/2! + 1/3! + 1/4! + \ldots$

$= 2.718281828459 \ldots$

(e is known to more than 1,000,000 decimal places.)

Furthermore, if $-1 < x \leqslant 1$,

(2) $\log_e(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + - \ldots$

If n is an integer $\geqslant 1$, then (putting $x = 1/n$)

$$\log_e(1 + 1/n) = 1/n - 1/2n^2 + 1/3n^3 - + \ldots,$$

so $$\log_e(1 + 1/n)^n = n \log_e(1 + 1/n)$$
$$= 1 - 1/2n + 1/3n^2 - + \ldots$$

so $$\log_e[(1 + 1/n)^n/e] = -1/2n + 1/3n^2 - + \ldots$$

so $$(1 + 1/n)^n/e = e^{(-1/2n + 1/3n^2 - + \ldots)}$$

This equation implies that as $n \to \infty$,

(3) $(1 + 1/n)^n/e \to 1.$

Moreover,

$$(1 + 1/n)^n/e = e^{-1/2n} \times e^{1/3n^2} \times e^{-1/4n^3} \times \ldots$$

$$= (1 - 1/2n + 1/8n^2 - 1/48n^3 + \ldots) \quad \text{(using (1))}$$

$$\times (1 + 1/3n^2 + 1/18n^4 + \ldots)$$

$$\times (1 - 1/4n^3 + \ldots)$$

$$\times \ldots$$

$$= 1 - 1/2n + 11/24n^2 - + \ldots,$$

so

(4) $(1 + 1/n)^n = e(1 - 1/2n + 11/24n^2 - + \ldots).$

In particular, if we put $n = 1000$, we obtain

$$(1.001)^{1000} = e(1 - 1/2{,}000 + 11/24{,}000{,}000 \ldots)$$

$$= 2.718281828459 \ldots \times 0.999500458 \ldots$$

$$= 2.716924\ldots$$

so $1001^{1000} = 10^{3000} \times 2.716924 \ldots$.

In fact, using the computer in the school of mathematics at U.N.S.W. that first year students learn on, it is easy to find all the digits of 1001^{1000}.

$1001^{1000} =$

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