COMMENTS ON THE 1978 SCHOOL MATHEMATICS COMPETITION

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The standard of performance in the 1978 School Mathematics Competition was much the same as it has been for some years, which, apart from the top sixty or so in each division is, frankly, pretty poor. This year, the top student in each division answered about 3½ of the 5 questions, but the vast majority could manage only 1 question. It is a fact however, that, as usual, every question was answered well by someone.

I feel that many of the entrants in the competition have the wrong idea. The problems are not likely to be textbook-type-problems-only-a-bit-harder. They are, generally speaking, quite different. They require persistence or ingenuity or both. They require the competitor to "muck around", to experiment, to investigate, to discover!

The competition is aimed at those school-students who have, or are suspected of having, some ability at tackling problems of this nature.

The competition is an "open-book" exam. The competitor is allowed to take in with him/her any books, calculators, aids, of any sort. But from what I said before, it is unlikely that the school text will be of any use. Rather, problem books or books about mathematics are relevant.

One of the best ways of preparing for the competition is to tackle problems from past years. These together with solutions are to be found in Parabola No. 2 for each year since 1972 (see the Parabola subscription form).

Having made these comments, let me wish entrants in the 1979 competition success.

Now I will discuss some of the problems, and their solutions, in the 1978 competition (see Vol. 14 No. 2).

Junior Division

1. **(i)** All that is required here is a list of the first few primes and a little "trial-and-error".

(ii) J. Mastorides of Caringbah High pointed out that there is a set of five different primes such that the sum of every three is also prime if we allow negative primes, that is, if $-p$ is prime whenever p is. (This is the usual definition in some contexts, but we didn't think of it in setting the problem.) Thus $\{-11, -5, 19, 23, 29\}$ is such a set.

2. Let $a(n) = (n!)^{1/n}$.

Some students realised that you were asked to show that the sequence

$$a(1),\ a(2),\ a(3),\ \ldots$$

is an increasing sequence. Writing

$$a(n) = b^{(\log 1 + \log 2 + \ldots + \log n)/n}$$
$$= b^{\text{average}(\log 1, \log 2, \ldots, \log n)}$$

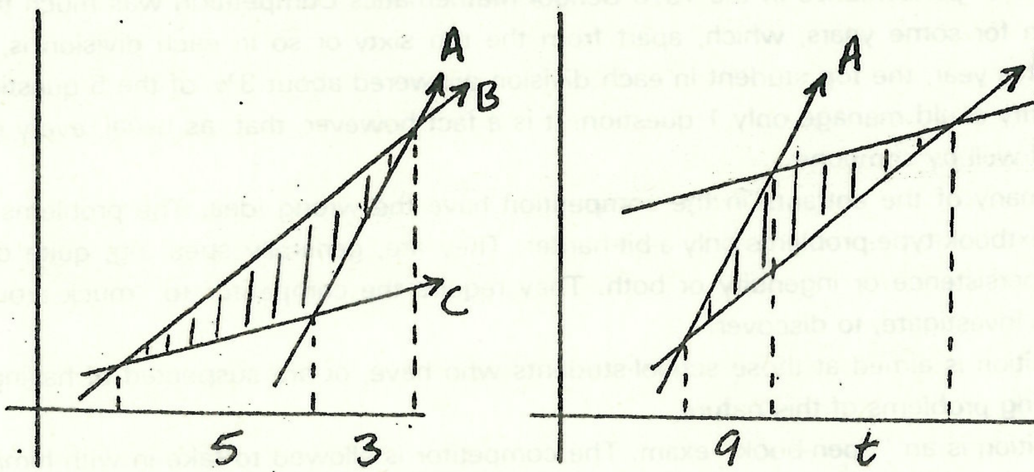
where b is any base (e.g. 10), one sees clearly that a(n) increases with n.

Some students noticed that $a(n+1) - a(n)$ is roughly constant. Indeed it is true, but hard to prove, that

$$a(n+1) - a(n) \to e^{-1} = 0.3678794\ldots.$$

(See "Try This for Size" for more about e.)

4. A few students realised that they could use travel-graphs to great advantage here.



The two shaded triangles are similar. It follows that t:9 = 5:3, so t = 15.

Senior Division

2. (i) All that was needed here was a little persistence. There are shortcuts which final-year students should know. Thus for example, the number of ways of obtaining the arrangement 3, 2, 1 is

$${}^6C_1 \times {}^5C_1 \times {}^4C_1 = 6 \times 5 \times 4 = 120$$

(The number of choices for the alley containing the 3 balls is ${}^6C_1$. There are then five alleys from which to choose the alley for the 2 balls, giving ${}^5C_1$ choices, leaving four alleys for the single ball, giving ${}^4C_1$ choices.)

Thus the complete list of arrangements is

$$6 \quad {}^6C_1 = 6 \text{ arrangements}$$
$$5,1 \quad {}^6C_1 \times {}^5C_1 = 30$$
$$4,2 \quad {}^6C_1 \times {}^5C_1 = 30$$

4,1,1	$^6C_1 \times {}^5C_2 = 60$	
3,3	$^6C_2 = 15$	
3,2,1	$^6C_1 \times {}^5C_1 \times {}^4C_1 = 120$	
3,1,1,1	$^6C_1 \times {}^5C_3 = 60$	
2,2,2	$^6C_3 = 20$	
2,2,1,1	$^6C_2 \times {}^4C_2 = 90$	
2,1,1,1,1	$^6C_1 \times {}^5C_4 = 30$	
1,1,1,1,1,1	$^6C_6 = 1.$	

2. **(ii)** Here one has to count the number of arrangements of 5A's and 1B, 4A's, 1B and 1 C, and of 2A's, 2B's and 2C's in a line.

Thus the number of arrangements of 5A's and 1B is $^6C_5 \times {}^1C_1 = 6$. (There are $^6C_5$ choices of the five spots for the A's, leaving $^1C_1$ choices for the B.) Similarly the number of arrangements of 4A's, 1B and 1C is $^6C_4 \times {}^2C_1 \times {}^1C_1 = 30$ and of 2A's, 2B's and 2C's is $^6C_2 \times {}^4C_2 \times {}^2C_2 = 90$.

This completes the solution. However, it might be nice to have a complete list of all the probabilities:

Pr(6)	$= 6 \times Pr(A^6)$	$= 6 \times$	$1/6^6 =$	$6/6^6$
Pr(5,1)	$= 30 \times Pr(A^5B^1)$	$= 30 \times$	$6/6^6 =$	$180/6^6$
Pr(4,2)	$= 30 \times Pr(A^4B^2)$	$= 30 \times$	$15/6^6 =$	$450/6^6$
Pr(4,1,1)	$= 60 \times Pr(A^4B^1C^1)$	$= 60 \times$	$30/6^6 =$	$1800/6^6$
Pr(3,3)	$= 15 \times Pr(A^3B^3)$	$= 15 \times$	$20/6^6 =$	$300/6^6$
Pr(3,2,1)	$= 120 \times Pr(A^3B^2C^1)$	$= 120 \times$	$60/6^6 =$	$7200/6^6$
Pr(3,1,1,1)	$= 60 \times Pr(A^3B^1C^1D^1)$	$= 60 \times$	$120/6^6 =$	$7200/6^6$
Pr(2,2,2)	$= 20 \times Pr(A^2B^2C^2)$	$= 20 \times$	$90/6^6 =$	$1800/6^6$
Pr(2,2,1,1)	$= 90 \times Pr(A^2B^2C^1D^1)$	$= 90 \times$	$180/6^6 =$	$16200/6^6$
Pr(2,1,1,1,1)	$= 30 \times Pr(A^2B^1C^1D^1E^1)$	$= 30 \times$	$360/6^6 =$	$10800/6^6$
Pr(1,1,1,1,1,1)	$= 1 \times Pr(A^1B^1C^1D^1E^1F^1)$	$= 1 \times$	$720/6^6 =$	$720/6^6$

[Total = $(6 + 180 + 450 + 1800 + 300 + 7200 + 7200 + 1800 + 16200 + 10800 + 720/6^6 = 1.$]

4. This question is closely bounded up with Cardan's solution of the cubic equation. It was disappointing that only one or two students had read enough about cubics to be able to handle this problem. (The Encyclopedia Britannica is a surprising but useful reference here.)

Actually, what the published solution shows is that the inverse of the (monotonic, hence invertible) function

$$f(x) = \tfrac{1}{2}(x^3 + 3x)$$

is

$$f^{-1}(x) = \sqrt[3]{(x + \sqrt{(x^2 + 1)})} + \sqrt[3]{(x - \sqrt{(x^2 + 1)})}.$$

5. This was one of the rare "applied" problems we have in the competition, and the second part was poorly done.

I must apologise for the inaccuracy in the numerical answer — I used a slide rule (and badly). In fact,

$$\sin\phi = \sqrt{(\sin^2 43\tfrac{1}{2}^\circ + \cos 52^\circ \cos 35^\circ \sin^2 76^\circ)}$$

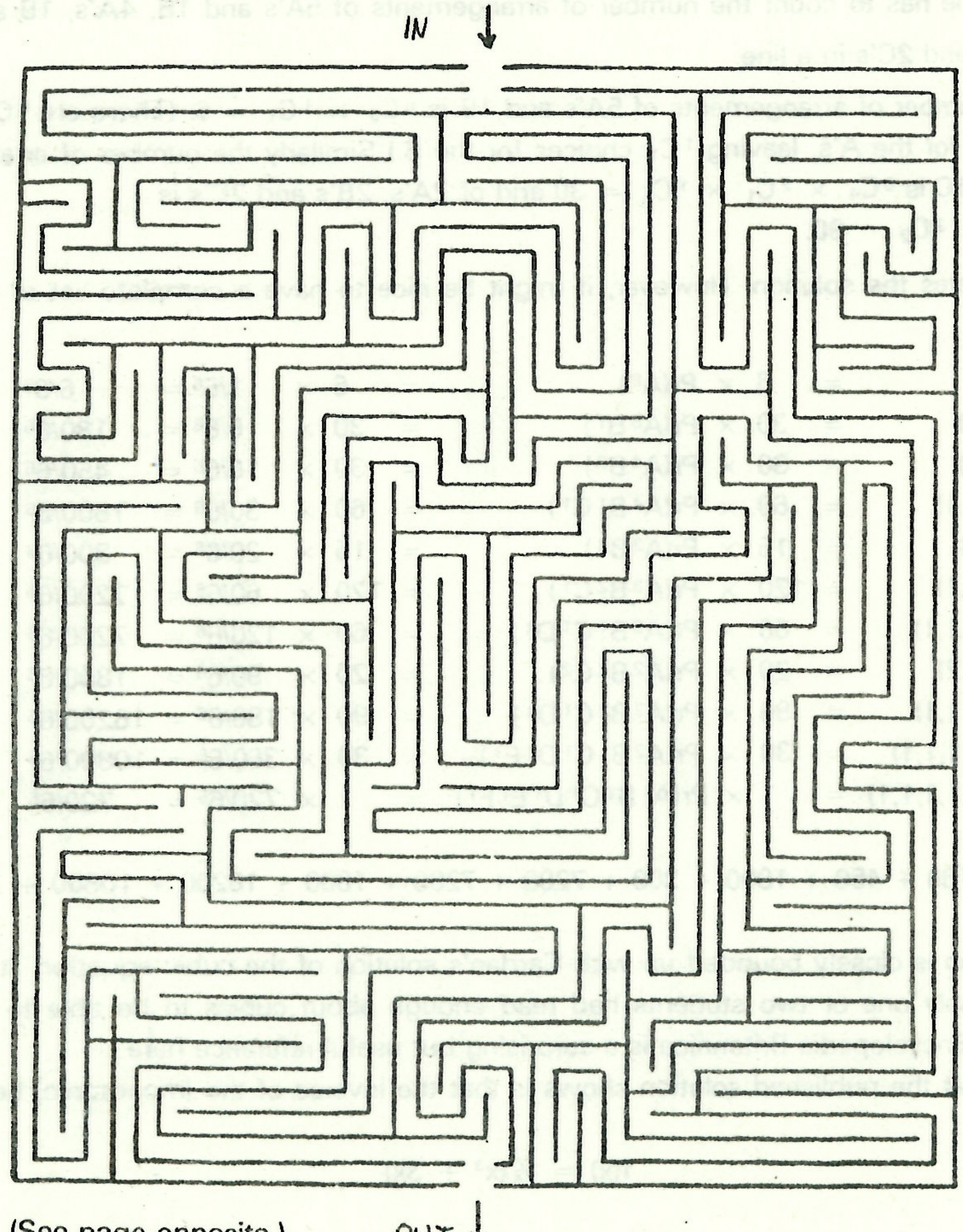
$$\cong 0.974,$$

$$\phi \cong 76.9^\circ$$

$$\text{g. c. distance} = (2\phi)/(360^\circ) \times C$$

$$\cong 0.427 \times 40{,}000$$

$$\cong 17{,}090 \text{ km.}$$



(See page opposite.)