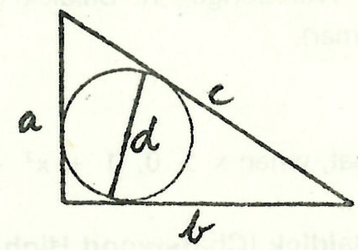
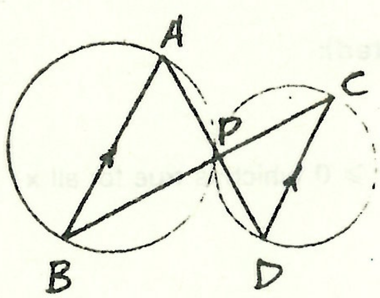
398. Show that it is impossible to construct an equilateral triangle on the pegboard in question 397 using 3 pegs and a rubber band.

399. Show how to construct an equilateral triangle by folding a single (rectangular) sheet of paper. No rulers, compasses or separate sheets for measuring are to be used.

400. Show that the diameter d of the inscribed circle of a right triangle of legs a, b and hypotenuse c satisfies $d = a + b - c$.





401. Let AB and CD be parallel diameters of two circles which touch at P. Show that the lines BC and AD intersect at P.

402. Consider 5 points in space such that each pair is not more than 1 cm apart. What is the greatest number of pairs which can be exactly 1 cm apart? Prove your answer. (If there are 4 points there can be as many as six pairs exactly 1 cm apart — take the four points at the vertices of a regular tetrahedron).

403. Given any set of ten distinct positive integers each less than 100 show that there are two subsets of this set having no elements in common such that the sums of the numbers in the subsets are equal.

404. The number 1234567 is not divisible by 11, but 3746512 is. How many different multiples of 11 can be obtained by appropriately ordering these digits?

Solutions to Problems from Vol. 14 No. 1

369. Find a five digit number which when divided by 4 yields another 5 digit number using the same 5 digits but in the opposite order.

Solution: G.J. Chappell (Kepnock High) writes:

Let the number be y = ABCDE and x = EDCBA so $y = 4x$. Now $y < 100{,}000$ so $x < 25{,}000$ so $E = 0, 1$ or 2 but y is even so $E = 0$ or 2, but x has five digits ($E \neq 0$) so $E = 2$.
Since $y = 4x$, $E = 4 \times A \pmod{10}$ so $A = 3$ or 8, but $x \geqslant 20{,}000$ so $y \geqslant 80{,}000$ so $A = 8$.
But $y \leqslant 89{,}992$ so $x \leqslant 22{,}498$; $x \geqslant 20{,}008$ so $D = 0, 1$ or 2. Since $y = 4x$, $D = 4 \times B + 3 \pmod{10}$ which is odd so $D = 1$.

$4B + 3 = 1 \pmod{10}$ so $B = 2$ or 7. $21{,}008 \leqslant x \leqslant 21{,}998$ so $84{,}032 \leqslant y \leqslant 87{,}992$ so $B = 7$.
21 C 78 × 4 = 87 C 12 and by considering the remainders carried over;
$4 \times C + 3 = 3C$ so because $4 \times 9 + 3 = 39$, $C = 9$.
The number is 87912.

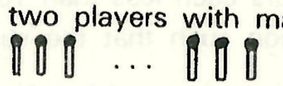
Correct solutions also from O. Wright (Davidson High), K. Svendsen (Busby High), P. Rider (St. Leo's College, Wahroonga), R. Baldick (Chatswood High), J. Taylor (Woy Woy High), P. Crump (Sydney Grammar).

370. Prove that, when $x > 0$, $(1 + x^2 + x^4)/(x + x^3) \geqslant 3/2$.

Solution: R. Baldick (Chatswood High) writes (somewhat abbreviated):
Lemma 1: $(1/x) + x \geqslant 2$ for all $x > 0$.
Lemma 2: $x/(x^2 + 1) \leqslant \frac{1}{2}$ for all $x > 0$.
After simplifying, each of these inequalities is equivalent to $x^2 - 2x + 1 \geqslant 0$ which is true for all x since $x^2 - 2x + 1 = (x - 1)^2$.

$$\begin{aligned} \text{Now } (1 + x^2 + x^4)/(x + x^3) &= (1 + 2x^2 + x^4)/(x + x^3) - x^2/(x + x^3) \\ &= ((1 + x^2)^2/x(1 + x^2)) - (x/(1 + x^2)) \\ &= ((1/x) + x) - x/(1 + x^2) \\ &\geqslant 2 - \tfrac{1}{2} \text{ using Lemmas 1 and 2.} \\ &= 3/2. \end{aligned}$$

Correct solutions also from K. Svendsen, P. Crump and G.J. Chappell, each of whom located the minimum of the given function or a closely related function by calculus.

371. A game is played by two players with matchsticks, as follows. To start, 36 matches are equally spaced in a row:  . Each player picks up in turn, either one, two, or three matches. The player who picks up the last match wins the game.

(a) Prove that the second player can always win.
(b) The rules are changed, to require that the one, two or three matches must be neighbouring matches from one group. Thus, after the first player's turn the remaining matches are in one or two groups. The second player must choose his matches, all together, from one of these groups. There will then be one, two or three groups, from one of which the first player must pick his matches, etc.
Can the second player still always win? Prove your answer.

Solution: O. Wright (reworded for brevity):
(a) The second player's winning strategy is to pick up $(4-n)$ matches in answer to the first player's n matches. In this way, the total number of matchsticks picked up after any turn of player 2 will be a multiple of 4. On his 9th turn, the second player will pick up the 36th matchstick.
(b) No. Consider the matchsticks in a line, labelled in order 1, 2, . . ., 36. The first player can always win by the following strategy:-

Move 1. He picks up the middle two matchsticks, 18 and 19. The remaining 34 matchsticks are now in two equal groups, A and B.

Later moves:- He adopts a symmetry strategy; he picks up the group of matchsticks that are an exact reflection of the matchsticks picked up by player 2. That is, for each matchstick labelled x picked up by player 2, player 1 picks up the matchstick labelled $37-x$. This always preserves the symmetry of what remains of group A and group B, and ensures that player 1 picks up the last match.

372. After the first day of classes, each of 5 different students knows a different bit of gossip about the teachers in their school. When they get to their separate homes, the telephoning begins. Assume that whenever anyone calls anyone else, each tells the other all the gossip he knows. What is the smallest number of calls after which it is possible for every student to know all 5 bits of gossip?

Solution by S.S. Wadhwa (Ashfield Boys' High):

Six calls are sufficient.

Say the five students are A; B; C; D; and E. The calls could be made in the following order:

(1) A to B (2) B to C (3) E to D (4) D to C (5) C to A (6) E to B.

Also solved correctly by P. Crump.

Other contributions: Several people who correctly solved more difficult problems persuaded themselves that a minimum of seven calls were required. However, K. Svendsen not only supplied the correct answer but showed that if S_n denotes the number of calls required for a similar problem with n people, then when n is even $S_n \leqslant n/2 + 2S_{n/2}$ and when n is odd $S_n \leqslant S_{n-1} + 2$, which with $S_1 = 0$ and $S_2 = 1$ enables an upper bound for S_n to be evaluated recursively.

373. For which values of n is $1^n + 2^n + 3^n + 4^n$ divisible by 5?

Solution: R. Baldick writes:

A number is divisible by 5 if it ends in a 0 or a 5. The table gives the last digits of 1^n, 2^n, 3^n, 4^n and the last digit of the sum for $n = 4k, 4k+1, 4k+2, 4k+3$. Since the last digits repeat, this table represents all possible combinations for positive integers n.

n =	4k	4k + 1	4k + 2	4k + 3
last digit of 1^n	1	1	1	1
last digit of 2^n	6	2	4	8
last digit of 3^n	1	3	9	7
last digit of 4^n	6	4	6	4
last digit of sum	4	0	0	0

(for k a non-negative integer and $n \neq 0$)

From the table it can be seen that $1^n + 2^n + 3^n + 4^n$ is divisible by 5 for positive n not a multiple of 4.

Correct solutions also received from S.S. Wadhwa, J. Taylor, D. Stickley (Woy Woy High), O. Wright, P. Crump and G.J. Chappell.

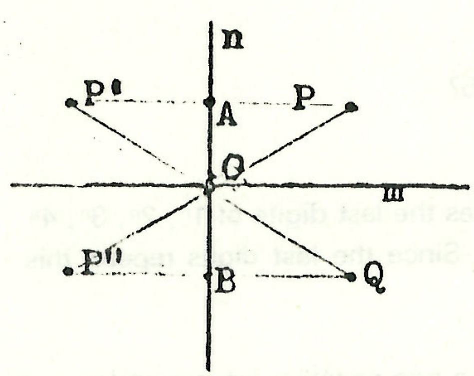
374. *(a)* A plane figure has one axis of symmetry and a point on that axis is a centre of symmetry. Does the figure necessarily have a second axis of symmetry?

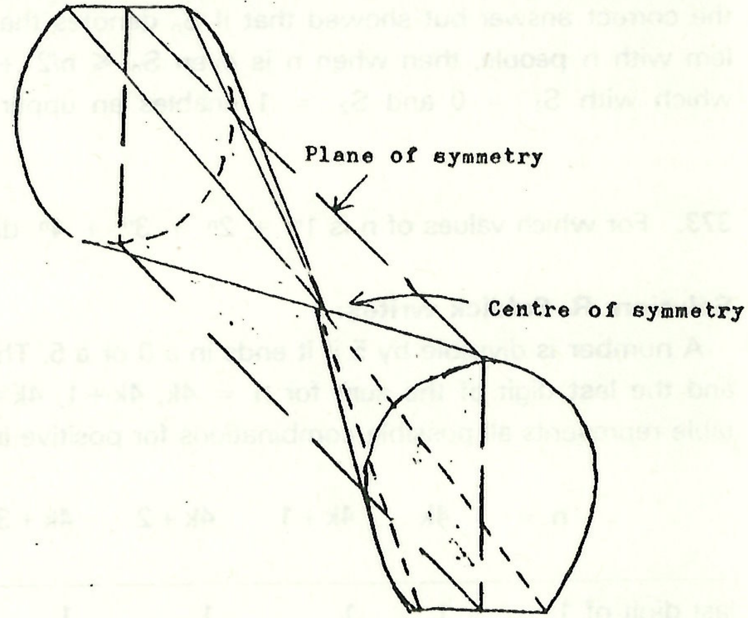
(b) A 3-dimensional figure has one plane of symmetry, and a point in that plane is a centre of symmetry. Does the figure necessarily have a second plane of symmetry?

Solution: R. Baldick writes:

(a) The figure does have a second axis of symmetry.

Proof: Let P be any point on the figure, n be the line of symmetry and O be the centre of symmetry on n. Since n is an axis of symmetry there exists a point P′, the reflection of P in n. Also there exists P″ the image of P through O. In addition there exists Q the reflection of P″ in n. By definition P′P and P″Q are perpendicular to n and bisected by it. Therefore $\triangle$P′AO is congruent to $\triangle$PAO, since AO is common to both, PA = P′A and PO = P′O. Similarly $\triangle$P″BO is congruent to $\triangle$QBO. But PO = P″O and PA = P″B by the symmetry of the figure. Therefore $\triangle$P′AO, $\triangle$P″BO are congruent and so AO = BO. Since P′P is parallel to P″Q, then a line m through O parallel to P′P will be equidistant from P′ and P″, and P and Q. Therefore Q is the reflection of P in m and P″ is the reflection of P′ in m. So m is the second axis of symmetry of the figure, since P can be any point on the figure.





(b) The 3-dimensional figure does not necessarily have a second plane of symmetry. Consider the illustrated 3-D figure which consists of 2 partial cones touching at their apices and with axes parallel. Clearly the figure has a centre of symmetry and exactly one plane of symmetry. Other similar figures are possible; they all show that the figure need not have a second plane of symmetry.

375. A cornfield has 1000 cornstalks. When the farmer stands at a cornstalk at the corner of the field he notices that some of the cornstalks line up with the one he is standing at. On closer examination it turns out that the number of these lines which contain an odd number (such as 1, 3, 5 . . .) of other cornstalks is odd. Is this true no matter which cornstalk he stands at? Why or why not?

Solution:

Yes. if he stands at cornstalk A, the 999 other cornstalks are partitioned into sets lying on straight lines through A. Now the sum of a collection of integers is even if there is an even number of odd summands, and odd if there is an odd number of odd summands. Hence to achieve the answer 999 there must be an odd number of the lines containing an odd number of other cornstalks.

376. Given n sacks each holding the same number of apples. On the first day an apple is removed from one sack. On the second day an apple is removed from each of 2 sacks; and so on until the nth day when one apple is removed from each of the n sacks.

The sacks are now all empty. For which n is this possible, and how is it to be done?

Solution: R. Baldick writes:

Each sack contains the same number of apples and there are n sacks.
In total $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$ apples are removed from the sacks.
Therefore n divides $\frac{1}{2}n(n+1)$.
Therefore $\frac{1}{2}(n+1)$ is an integer.
Therefore $n+1$ is even.
Therefore n is odd and there are $\frac{1}{2}(n+1)$ apples in each sack.

The apples may be removed by taking them out of the bags so that no bag has more than 1 apple more than any of the others. For example for $n = 5$, take 1 apple out of one of the bags, take 2 out of any 2 other bags. On the third day take 1 apple out of each of the sacks with 3 apples and another out of a sack with 2 apples. Then take 4 apples out of the remaining sacks which contains 2 apples. Finally take the last apple out of the 5 sacks.

Correct solutions also from J. Taylor and K. Svendsen.

377. Let x_i, y_i ($i = 1, 2, \ldots n$) be real numbers such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n$.

Prove that if $z_1, z_2, \ldots, z_n$ is any rearrangement of $y_1, y_2, \ldots, y_n$ then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \leqslant \sum_{i=1}^{n} (x_i - z_i)^2$$

Solution by D. Dowe (Geelong Grammar):

$$\sum_{i=1}^{n} (x_i - z_i)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} z_i^2 - 2 \sum_{i=1}^{n} x_i z_i .$$

Since the first two terms on the R.H.S. are the same for any arrangement of the z's the problem is equivalent to showing that $\sum_{i=1}^{n} x_i z_i$ is as large as possible if the z's are ordered (like the x's) in decreasing order of magnitude. We establish this by showing that if $j < k$ and $z_j < z_k$ then

$$\sum_{i=1}^{n} x_i z_i \leqslant \sum_{i \neq j,k} x_i z_i + x_j z_k + y_k z_j \qquad (1)$$

i.e. that the expression $\Sigma x_i z_i$ is not decreased by interchanging this pair of z's which is "out of order". Once this has been established the desired result is clear since the decreasing ordering of the z's is eventually obtained after repeated inversions of "out of order" pairs, and at no stage has the expression $\sum_{i=1}^{n} x_i z_i$ decreased.

To prove (1), suppose $z_k = z_j + \delta$ with $\delta > 0$.

$$\text{Then } \sum_{i=1}^{n} x_i z_i = \sum_{i \neq j,k} x_i z_i + x_j z_j + x_k z_k$$

$$= \sum_{i \neq j,k} x_i z_i + x_j z_j + x_k (z_j + \delta)$$

$$\leqslant \sum_{i \neq j,k} x_i z_i + x_j z_j + x_j \delta + x_k z_j \qquad \text{since } x_j \geqslant x_k$$

$$\leqslant \sum_{i \neq j,k} x_i z_i + x_j z_k + x_k z_j \text{ ; Q.E.D.}$$

Solution also received from K. Svendsen.

378. Let $a_1, a_2, a_3 \ldots$ be any infinite sequence of strictly positive integers such that $a_k < a_{k+1}$ for all k. Prove that infinitely many a_m can be written in the form $a_m = xa_p + ya_q$ with x, y strictly positive integers and $p \neq q$.

Solution:

This depends on the fact that if d is the g.c.d. of two integers a and b then there exist integers x and y such that $ax + by = d$. e.g. if $a = 5$, $b = 7$ then $d = 1$ and one pair of values for x, y is $x = 3$, $y = -2$. [This has been "bookwork" since the days of the ancient Greeks, being one of the important consequence of the Euclidean Algorithm. Other proofs are equally easy, but I will not present one here.]

It follows that any integral multiple of d can be expressed in the form aX + bY for appropriate integers X and Y. [a(kx) + b(ky) = kd]

If the number m = kd so represented is greater than ab where a and b are positive integers, one can find positive integers X_1, Y_1 such that $aX_1 + bY_1 = m$. To see this note that $m = aX + bY = a(X - cb) + b(Y + ca)$ where c is any integer. One can always choose c so that $0 < X - cb \leqslant b$ whence $a(X - cb) \leqslant ab$ and $b(Y + ca)$ must be positive. Now take $X_1 = X - cb$, $Y_1 = Y + ca$.

To summarise; if a and b are any two positive integers then all multiples of their g.c.d. d which are greater than ab are expressible in the form aX + bY with X, Y positive integers.

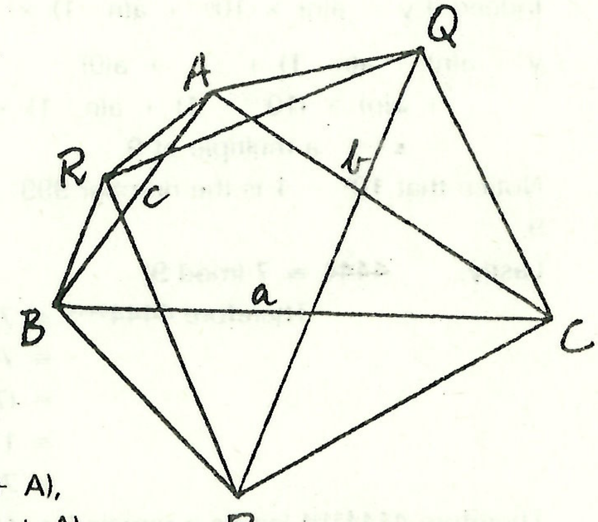
Now the integer a_1 has only a finite number of factors. Let d_n denote the g.c.d. of a_1 and a_n for every $n > 1$. Then at least one value of d_n must occur infinitely often. Suppose

$$d_{n_1} = d_{n_2} = \ldots = d_{n_k} = \ldots$$

$$k = 1, 2, 3, \ldots$$

Then each of $a_{n_2}, a_{n_3}, \ldots, a_{n_k}$ is a multiple of the g.c.d. of a_1 and a_{n_1}.
As all save possibly a few at the beginning of the list are greater than $a_1 \times a_n$, there are infinitely many values of n such that $a_n = a_1 X + a_{n_1} Y$ with X and Y positive integers.

379. On the sides of an arbitrary triangle ABC, triangles ABR, BCP and CAQ are constructed externally with $\angle PBC = \angle CAQ = 45°$; $\angle BCP = \angle QCA = 30°$; $\angle ABR = \angle RAB = 15°$. Prove that $\angle QRP = 90°$ and QR = RP.



Solution using trigonometry:

In $\triangle ARB$, by the sine rule,

$$AR/\sin 15° = c/\sin 150°$$

Therefore $AR = c/2 \cos 15°$.

In $\triangle AQC$, by the sine rule,

$$AQ/\sin 30° = b/\sin 105° = CQ/\sin 45°.$$

Hence $AQ = b/2 \cos 15°$ and $CQ = b/\sqrt{2} \cos 15°$.

Similarly applying the sine rule to $\triangle BPC$ yields

$$BP = a/2 \cos 15° \text{ and } CP = a/\sqrt{2} \cos 15°.$$

Now in $\triangle ARQ$, the cosine rule gives

$$QR^2 = AR^2 + AQ^2 - 2AR \times AQ \cos (60° + A),$$

whence
$$\begin{aligned} 4 \cos^2 15° \times QR^2 &= c^2 + b^2 - 2bc \cos (60° + A) \\ &= c^2 + b^2 - bc(\cos A - \sqrt{3} \sin A) \\ &= c^2 + b^2 - b(b - a \cos C) + \sqrt{3}\, ab \sin C \\ &= c^2 + ab \cos C + \sqrt{3}\, ab \sin C. \end{aligned} \quad (1)$$

Similar working in $\triangle BRP$ yields

$$4 \cos^2 15° \times PR^2 = c^2 + ab \cos C + \sqrt{3}\, ab \sin C \quad (2)$$

and in $\triangle CPQ$ yields

$$2 \cos^2 15° \times PQ^2 = b^2 + a^2 - ab \cos C + \sqrt{3}\, ab \sin C. \quad (3)$$

Comparing (1) and (2) gives PR = QR. Adding (1) and (2) gives

$$PR^2 + QR^2 = (2c^2 + 2ab\cos C + 2\sqrt{3}\,ab\sin C)/4\cos^2 15^\circ$$
$$= (c^2 + ab\cos C + \sqrt{3}\,ab\sin C)/2\cos^2 15^\circ$$
$$= ((a^2 + b^2 - 2ab\cos C) + ab\cos C + \sqrt{3}\,ab\sin C)/2\cos^2 15^\circ$$
$$= (a^2 + b^2 - ab\cos C + \sqrt{3}\,ab\sin C)/2\cos^2 15^\circ$$
$$= PQ^2.$$

Thus, by Pythagoras' Theorem, $\triangle PQR$ is right-angled at R.

380. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

Solution:

$$4444^{4444} < 10{,}000^{4444} = 10^{4\times 4444} = 10^{17{,}776}.$$

Hence 4444^{4444} has fewer than 17,777 digits.

Therefore $A < 9 \times 17{,}776 < 199{,}999$

and $B \leqslant 46$.

Therefore the sum of the digits of B is less than or equal to 12 (since of all integers less than or equal to 46, that for which the sum of the digits is largest is 39).

Next observe that if x is the sum of the digits of y, then both x and y leave the same remainder on division by 9. (*)

Indeed if $y = a(n) \times 10^n + a(n-1) \times 10^{n-1} + \ldots + a(0)$, then

$$y = a(n) + a(n-1) + \ldots + a(0)$$
$$+ a(n) \times (10^n - 1) + a(n-1) \times (10^{n-1} - 1) + \ldots + a(1) \times (10 - 1)$$
$$= x + \text{a multiple of } 9.$$

Notice that $10^k - 1$ is the number $999\ldots9$ with k digits all equal to 9, so is certainly divisible by 9.

Lastly, $4444 \equiv 7 \pmod 9$

Therefore $4444^{4444} \equiv 7^{4444} \pmod 9$
$$\equiv 7^{3\times 1481+1} \pmod 9$$
$$\equiv (7^3)^{1481} \times 7^1 \pmod 9$$
$$\equiv 1^{1481} \times 7 \pmod 9$$
$$\equiv 7 \pmod 9)$$

Therefore 4444^{4444} leaves a remainder of 7 on division by 9.

By the remark (*), A, B, and finally the sum of the digits of B all leave remainder 7 on division by 9. But since this last number is not more than 12, it is easy to check that it must be precisely equal to 7.

Correct solutions from O. Wright, S.S. Wadhwa, D. Dowe.