THE DEVIL'S DICE

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I cannot believe that God plays dice with the cosmos
—Einstein

The Puzzle

Once upon a time, in a kingdom beset by evil influences, a benevolent king condemned his court sorcerer to death for casting evil spells and being in league with the devil. However, the sorcerer asked for one last wish and, with a puff of smoke, he produced four large dice with each of their faces coloured one of the four colours, red, green, blue or white. "Sire," he addressed the king, "before you the dice are stacked squarely on top of each other in a column so that down any side of the column you see each of the four colours exactly once. My wish is to live until you can reassemble the dice in that way." With that, the sorcerer mixed up the dice and handed them to the king. Ever since that time, the four cubes have been known as the Devil's Dice.

You can make a set of the dice by enlarging Figure 1 on cardboard so that each cell is, say, one inch square, and colouring the faces as indicated on the pattern with R red, G green, B blue and W white. Then cut out along the heavy lines, bend along the dotted lines to form cubes and, lastly, stick each cube together with transparent tape. The cubes are numbered for easy reference later.

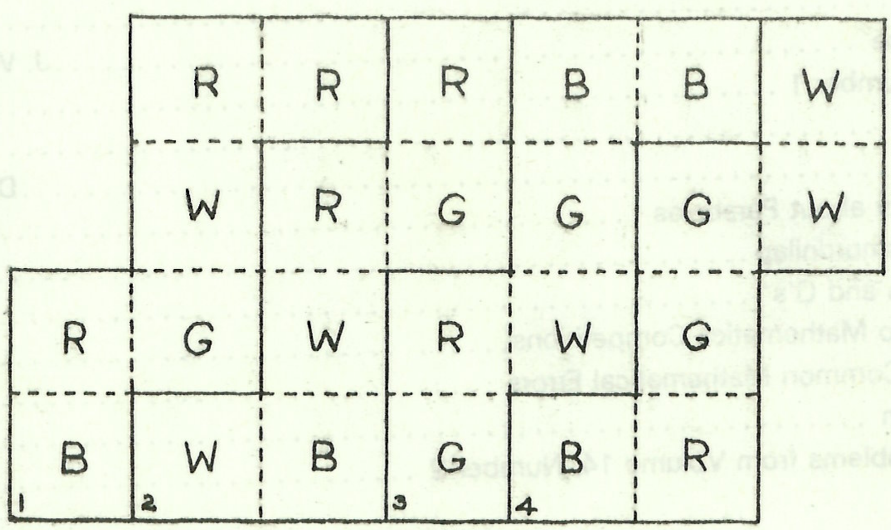

Figure 1

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Could you solve the puzzle and rid yourself of the evil sorcerer? For a start, we might proceed by trial and error. Let us see how many possible arrangements there are. First, there are three different ways of building a column onto the first cube: we have to choose one pair of faces as the top and bottom, but it doesn't matter which one we call top. Then, in stacking each subsequent cube, there are 24 possible orientations: there are six choices for the face to be joined to the previous cubes and then four possible rotations before the position of the new cube is fixed. In all, this gives $3 \times 24 \times 24 \times 24 = 41472$ arrangements of the cubes. So, should the king use trial and error, even if he could keep track of the arrangements already tried, the sorcerer could well live to a ripe old age. We will see, in the remainder of this article, how mathematics can triumph over evil.

The Graph

The key to the solution is a profoundly simple observation: when one face of a cube is lined up, the face on the opposite side of the cube is automatically fixed as well. Consequently, the faces of the cubes should be considered in opposite pairs. This information is recorded in Figure 2, which we shall call the graph of the dice. First, draw four circles, labelled R, G, B, and W, to represent the four colours. Now, the first cube has green and red faces opposite each other, so we draw a line labelled 1 between G and R. Similarly, there is a line labelled 1 between B and W and a loop labelled 1 at R. (The loop represents the pair of opposite red faces.) We do the same for the other three cubes. For example, a line labelled 3 connects B and R because the third cube has blue and red faces opposite each other.

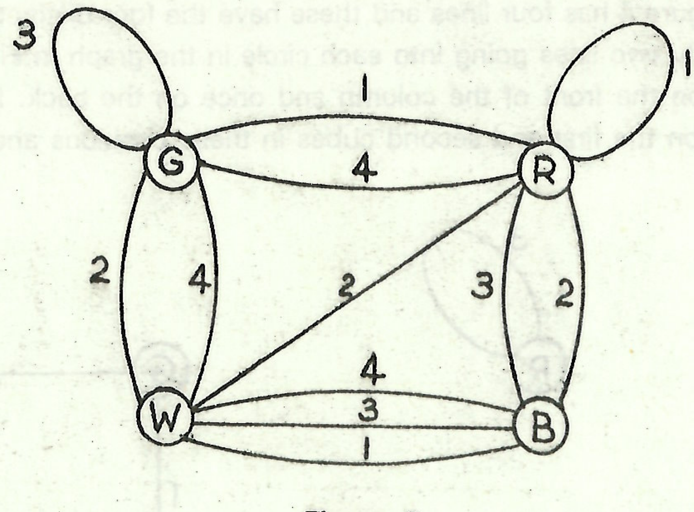

Figure 2

How can the graph in Figure 2 be used to solve the puzzle? So as not to give the solution away, let us consider a different set of four cubes whose graph is the one in Figure 3. This graph does actually come from a set of cubes. Can you build them?

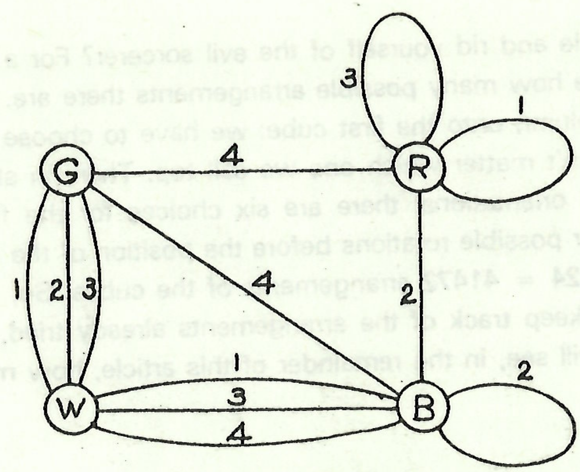

Figure 3

Imagine that the puzzle represented in Figure 3 stands solved on the table with the first cube on the bottom and the second, third and fourth in order going up the column. Each of the four colours occurs exactly once down each side of the column. Consequently, every colour occurs exactly twice on the eight faces consisting of the four faces down the front of the column and their four opposite faces down the back of the column. We can draw a graph with four lines to record the four pairs of opposite colours. For example, if the front and back faces of the first cube are blue and white, those of the second cube are green and white, those of the third cube are both red and those of the fourth cube are blue and green, our graph will be the one in Figure 4. In any case, this graph must have the following three properties:

(i) The graph in Figure 4 is a subgraph of the complete graph of the dice in Figure 3, that is every line in Figure 4 occurs in Figure 3 with the same label. This must be the case because Figure 3 records the colours on all the pairs of opposite faces of the cubes.

(ii) The graph in Figure 4 has four lines and these have the four distinct labels 1, 2, 3 and 4.

(iii) There are exactly two lines going into each circle in the graph in Figure 4 because each colour occurs just once on the front of the column and once on the back. For example, in Figure 4, there are white faces on the first and second cubes in these positions and there are two red faces on the third cube.

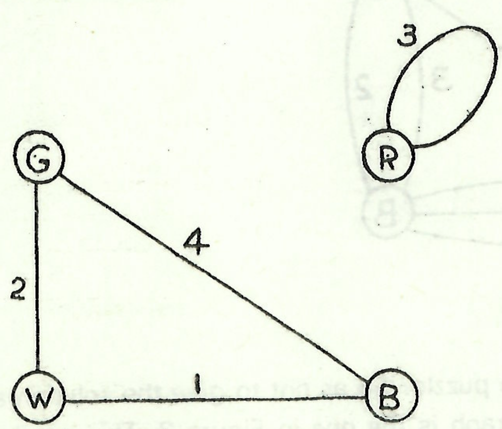

Figure 4

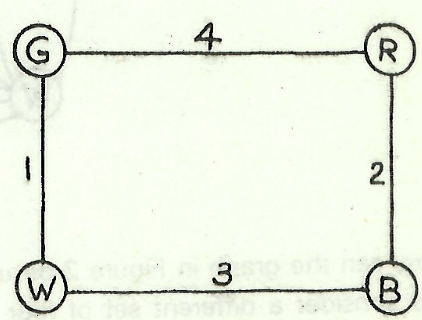

Figure 5

Returning to our imaginary solution of the puzzle, we can carry out the same procedure with the four faces on the left of the column and their opposite faces on the right. This gives another subgraph of the graph in Figure 3, satisfying properties (i), (ii) and (iii); it might be the one in Figure 5. The two subgraphs obtained this way have no lines in common because they refer to different pairs of opposite faces on the cubes.

The Solution

So far, we have imagined the puzzle to be solved and found two subgraphs of the graph in Figure 3 which have no line in common and satisfy (i), (ii) and (iii) above. Conversely, given two such subgraphs, say the ones in Figure 4 and 5, it is easy to solve the problem represented by the graph in Figure 3. Take the first cube. Figure 4 tells us that the opposite blue and white faces must point towards and away from us. Fix these faces between thumb and forefinger and rotate the cube about this axis. Figure 5 tells us that the opposite green and white faces must face left and right, however, we can't tell yet whether green should be to the left or to the right. Choose one orientation and put the first cube on the table, remembering that we might have to turn it upside down later. Now go through the same process with the second cube and put it on top of the first cube. If there is a repeated colour down one of the columns, flip one of the cubes over to interchange either the front and back or the left and right faces. Property (iii) guarantees that this process of flipping can be used to remove all the unwanted repetitions of colours. All this is much easier to do than to describe. Do the same for the third and fourth cubes and the puzzle is solved!

This method is all very well provided that subgraphs with the right properties can be found. Perhaps these are no easier to determine than a solution to the original problem by trial and error. In fact, this is where the graphical method saves energy. Regardless of what the original graph of the puzzle may be, the only possible subgraphs satisfying (ii) and (iii) above are the ones in Figure 6. To solve the puzzle, we only have to check through those subgraphs of the graph of the puzzle which appear in Figure 6 and to choose two which have no line in common.

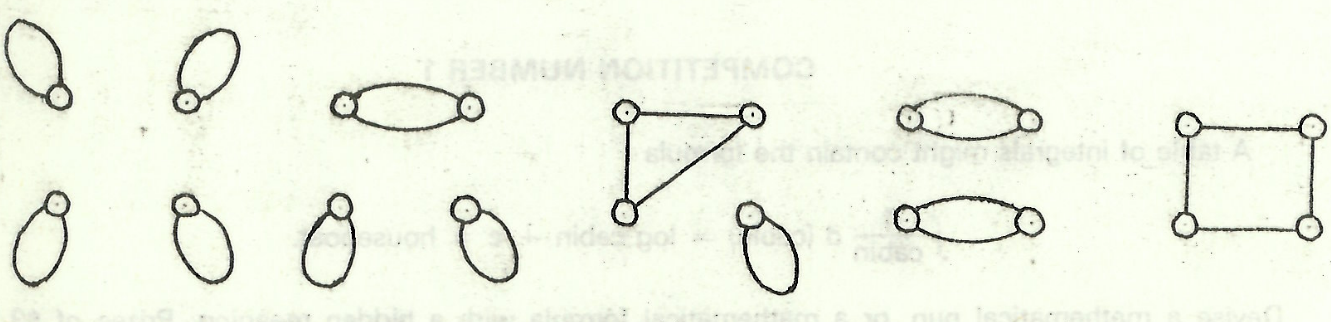

Figure 6

Now, apply the method to Figure 2 and triumph over the Devil's Dice. Actually, the method can provide a lot more information. Since different pairs of subgraphs give different solutions, we can quite easily calculate the number of different solutions to our puzzle. How many solutions are there for the Devil's Dice of Figure 1 and 2? What about Figure 3? By reversing the solution procedure, we can design new puzzles: label two of the graphs of Figure 6 with colours and numbers, put them together and add four more lines numbered 1, 2, 3 and 4. The resulting graph corresponds to a puzzle which is guaranteed soluble! Can you design a hard puzzle, that is one with only a small number of solutions? Of course, if we colour the cubes at random, there may very well be no solution to the puzzle, even if each colour appears at least once on each cube. Can you give an example of an impossible puzzle?

References

Many books on graph theory have accounts of puzzles like the Devil's Dice. One example is "Finite Graphs and Networks: An Introduction with Applications" by Busacker and Saaty (McGraw-Hill, 1965).

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COMPETITION NUMBER 1

A table of integrals might contain the formula

$$\int \frac{1}{\text{cabin}}\, d\,(\text{cabin}) = \log \text{cabin} + c = \text{houseboat}.$$

Devise a mathematical pun, or a mathematical formula with a hidden meaning. Prizes of \$3, \$2 and \$1 will be awarded for the best entries. Entries must reach the Editor by 1 July 1979 (not, you will notice 1 April) and the prize-winners will be announced in Parabola, Volume 15, Number 3.

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