MAGIC STARS

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If you look at Figure 1, you will see stars. Stars come in various sizes: the figure shows the five-pointed star, or pentagram, which was the secret symbol of Pythagoras, the six-pointed star, or star of David, the seven-pointed star and the eight-pointed star, or octopus. To draw an n-pointed star, take n points uniformly spaced round a circle and join every second point. As you can see from the figure, one of two things can happen. If n is odd, the star is actually a polygon with n vertices and n sides enclosing the centre of the circle twice. (The points inside the circle where the sides cross are not counted as vertices here.) On the other hand, if n is even, the star consists of two separate polygons each with ½n sides. Here are some starred questions: Is there such a thing as a four-pointed star? What happens if you vary the construction and join every third point round the circle? Do stars really have points?

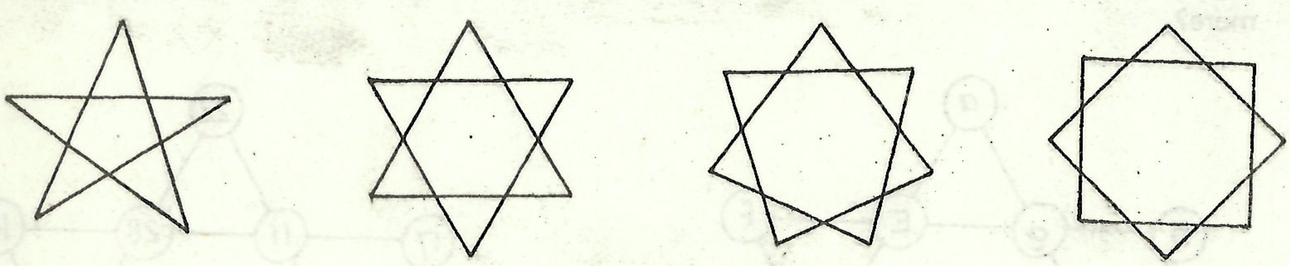

Figure 1

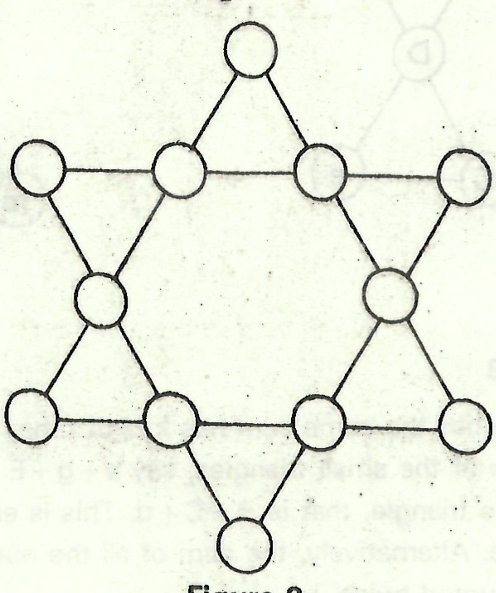

Figure 2

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We are interested here in a problem based on the star of David. Draw circles, as shown in Figure 2, at each point of intersection of two of the sides of the star. This gives twelve circles arranged in six lines with four circles on each line. The problem is to place a number in each circle so that the numbers in each of the six lines have the same sum.

There is one obvious solution to the problem: put the same number in each circle! To make the problem a little more interesting (and difficult), we shall insist that the numbers in the star are all different. To construct such a star, we might try to fill in the circles one by one. (See Figure 3.) We can apparently fill in the entries a, b, c, d, e, f and g at will, since they do not form any complete lines. Let us call the sum of each line in the star s and label the remaining circles A, B, C, D, and E. Then, working round the star, we find $A = s-a-c-g$ to make the line joining circles a and c add to s, $B = a-b+c-d+g$, $C = s-a+b-2c+d-e-g$, $D = a-b+2c-2d+e-f+g$ and, finally, since the circle E lies on two lines,

$$E = s-b-f-g = s-2a+b-2c+2d-2e+f-g.$$

Consequently, if we choose

$$a+c+e = b+d+f, \qquad (\star)$$

we get a star with each line having the sum s. Now it only remains to choose the parameters satisfying ($\star$) in such a way that the entries in the circles are distinct positive integers. This needs a little experimentation, but is not too difficult; one solution is shown in Figure 4. Can you find some more?

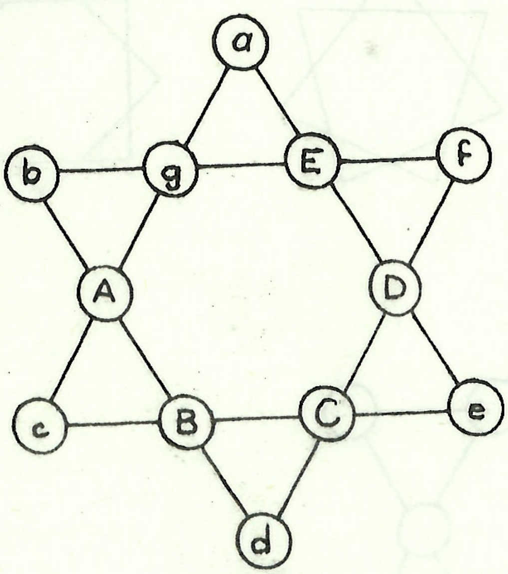

Figure 3

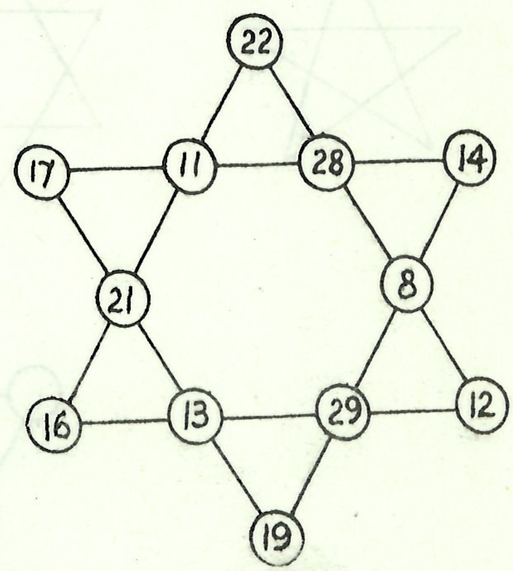

Figure 4

A star in which each line has the same sum has lots of other magic properties. For example, the sum of the numbers in one of the small triangles, say $a+g+E$ in Figure 3, is equal to the sum of the numbers in the opposite triangle, that is $B+C+d$. This is easy to check using the expressions found for B, C and E above. Alternatively, the sum of all the numbers on the three lines bf, ac and ae, with common points counted twice, is

$$2a + 2g + 2E + b + c + e + f + A + D,$$

while the sum of all the numbers on the three lines ce, bd and df is

$$2d + 2B + 2C + b + c + e + f + A + D.$$

Since each line has the same sum, these last two expressions are equal and so $a+g+E = d+B+C$, as required. (This time, we have only used the letters in Figure 3 as convenient labels; we do not need to use our previous calculations.) By symmetry, this property holds for each of the three pairs of opposite triangles in the star. Can you find further magic properties of the six-pointed magic star by taking different sums of lines? Can you find any similar results for n-pointed magic stars?

As we saw above, it is quite easy to construct galaxies of six-pointed magic stars. Here is a more challenging problem: Can you construct a six-pointed magic star using the numbers 1, 2, 3, . . ., 12 to fill in the circles? This calls for a little enlightened experimentation and the property of the opposite triangles should be of some use in narrowing down the possible configurations.

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The starry-eyed reader will find more tantalising magic in Martin Gardner's column in Scientific American, December 1965, and in Henry Ernest Dudeney's "Puzzles and Curious Problems" (Fontana, 1967).

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ELEVENIFICATION

Ian Black of Sydney Grammar School has sent in the following problem. If we reverse the digits of the number 69 and add the result to the original number, we get 69 + 96 = 165, which is divisible by 11. This does not work for 134, since 134 + 431 = 565 is not divisble by 11, but if we repeat the process we get 565 + 565 = 1130 and then, again, 1130 + 0311 = 1441, which is divisible by 11. Can any number be converted to a number divisible by 11 if we carry this process of reversal and addition on for long enough?

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