THE INSIDE STORY ABOUT PARABOLAS

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What is a parabola?

According to one small dictionary (brand-name suppressed), it is "the curve formed by the intersection of a cone with a plane parallel to its side." This definition is illustrated in Figure 1 which shows a cone C and a plane P cutting the cone and "parallel" to the side OA. The curve of intersection I is a parabola. If we look at the mid-section OAB of the cone, we see Figure 2; the plane P is perpendicular to the page. This is what is meant by the rather vague use of "parallel" above. Another way to see this construction is to shine a torch with a conical reflector against a wall. As you change the angle between torch and wall, you will see all three types of conic sections – ellipses, parabolas and hyperbolas. In fact, this is the way the ancient Greeks defined the parabola.

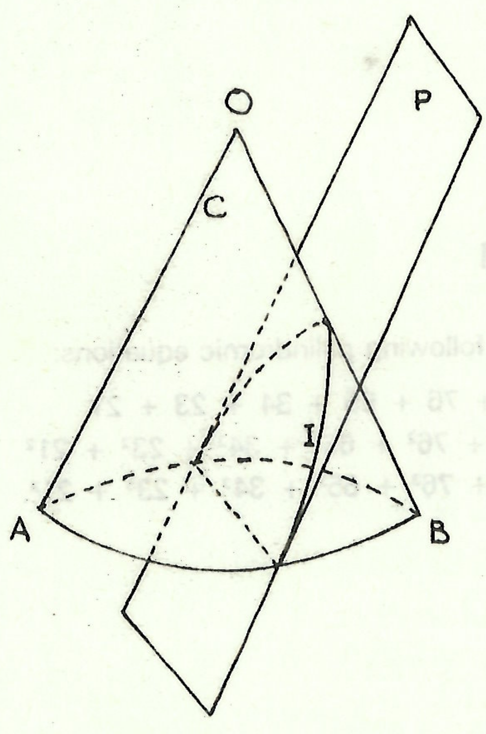

Figure 1

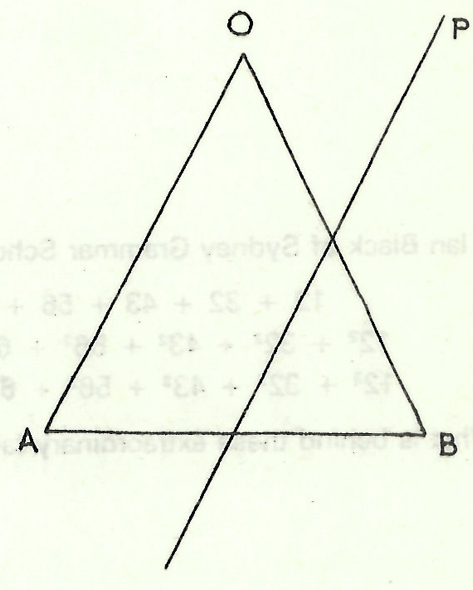

Figure 2

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The same dictionary goes on to say that a parabola is also "the locus of points which are equidistant from a given point F, the focus, and a given line D, the directrix". That is, in Figure 3, the distance XF from a point X on the parabola to F is the same as the perpendicular distance XY from X to the line D. We can easily derive the familiar equation of the parabola from this definition. We draw x- and y-axes as in Figure 3 so that the focus F of the parabola is at (a,0) and the directrix is the line $x = -a$. Let X with coordinates (x,y) be a point on the parabola. The two distances we need are

$$XF = \sqrt{[(x-a)^2 + y^2]} \text{ and } XY = x + a.$$

These two distances are equal: XF = XY. Squaring both sides gives $XF^2 = XY^2$,

$$(x-a)^2 + y^2 = (x+a)^2,$$

and a little simplification gives the familiar equation

$$y^2 = 4ax. \tag{1}$$

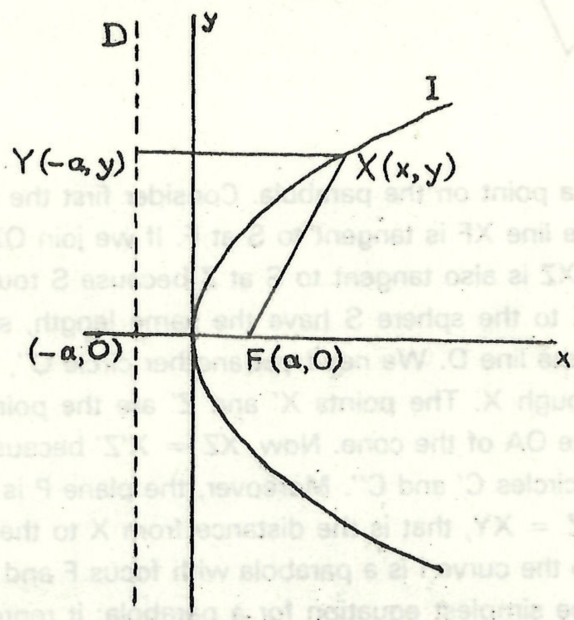

Figure 3

Now we have two definitions of a parabola. Are they consistent? Figure 4 is not a drawing of the nose-cone of a discarded Sputnik; it is, in fact, the key to answering our question. We start with Figure 1 and draw a sphere S which touches the sides of the cone along the circle C′, say, and touches the plane P at the point F. The circle C′ lies in a "horizontal" plane H and the two planes P and H meet in a line D. Figure 5 shows all this in section. The idea, not too surprisingly, is to show that F is the focus and D is the directrix of our parabola I. The ingenious construction in Figure 4 was discovered in 1822 by Germinal Dandelin.

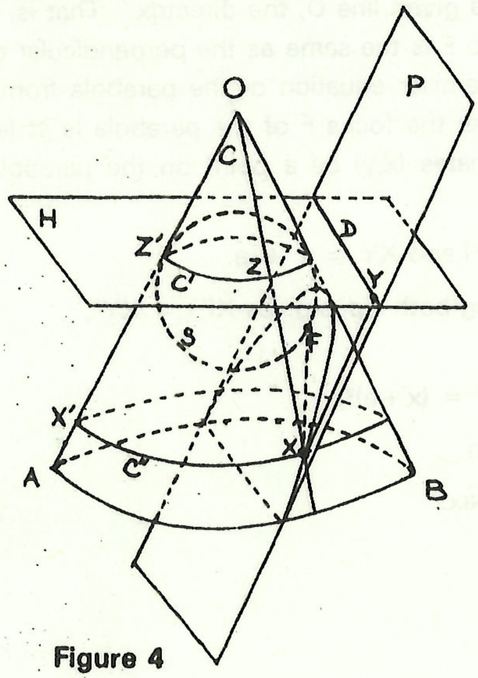

Figure 4

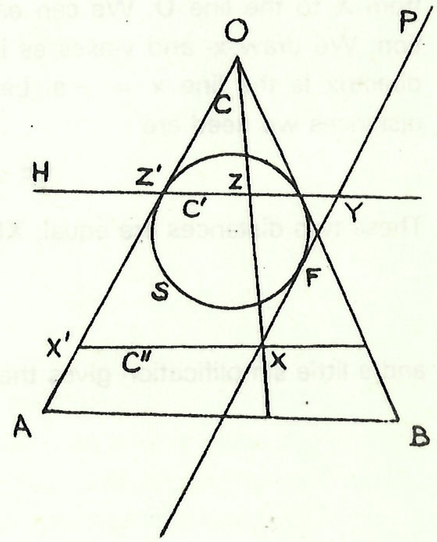

Figure 5

Here goes. Let X be a point on the parabola. Consider first the distance XF. Since the sphere S touches the plane P, the line XF is tangent to S at F. If we join OX and call Z the point where OX cuts the circle C′, then XZ is also tangent to S at Z because S touches the cone along C′. Now all tangents drawn from X to the sphere S have the same length, so XF = XZ. Next, consider the distance XY from X to the line D. We need yet another circle C″, the "horizontal" circle round the cone which passes through X. The points X′ and Z′ are the points where the circles C″ and C′ respectively cut the edge OA of the cone. Now, XZ = X′Z′ because these lines are both measured between the horizontal circles C′ and C″. Moreover, the plane P is parallel to the edge OA, so X′Z′ = XY. Hence XF = XZ = XY, that is the distance from X to the point F is equal to the distance from X to the line D. So the curve l is a parabola with focus F and directrix D.

The equation (1) is the simplest equation for a parabola; it represents a parabola with vertex at the origin and its axis along the x-axis, as in Figure 3. Let us investigate what happens to this equation when the parabola is in a different position.

What is the equation of a parabola with vertex V at the point (h,k) and axis parallel to the x-axis? To find this, we draw a new coordinate system with the origin at V and X- and Y-axes parallel to the old x- and y-axes, as shown in Figure 6. In the new coordinates, the equation of the parabola is $Y^2 = 4aX$, by (1). The relation between the two sets of coordinates is $X = x - h$, $Y = y - k$, as shown in Figure 6, so in the original coordinates, the equation of the parabola is

$$(y-k)^2 = 4a(x-h). \qquad (2)$$

So it is easy to move the vertex of the parabola away from the origin.

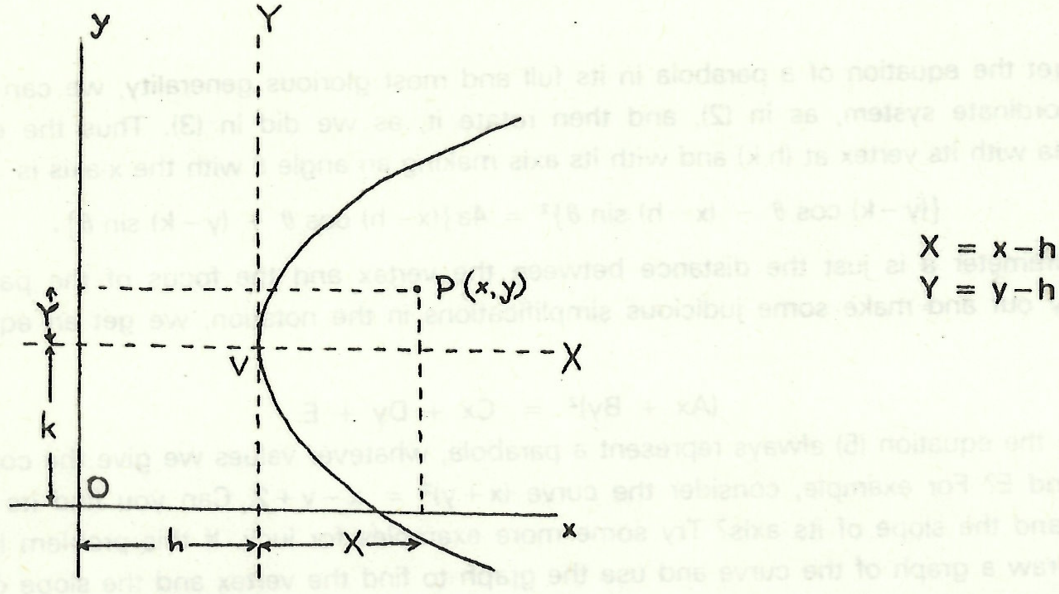

Figure 6

Now, let us work out the equation of a parabola with its vertex at the origin and with its axis making an angle θ with the x-axis. Here, we choose new X- and Y-axes by rotating the x- and y-axes through the angle θ. (See Figure 7.)

In the new coordinates, our parabola is again $Y^2 = 4aX$. To relate the two sets of coordinates, we take a typical point P in Figure 7 and observe that

$$x = OP\cos\phi, \quad y = OP\sin\phi$$

and

$$X = OP\cos(\phi-\theta) = OP\cos\phi\cos\theta + OP\sin\phi\sin\theta = x\cos\theta + y\sin\theta$$
$$Y = OP\sin(\phi-\theta) = -OP\cos\phi\sin\theta + OP\sin\phi\cos\theta = -x\sin\theta + y\cos\theta$$

So the equation of the parabola is

$$(y\cos\theta - x\sin\theta)^2 = 4a(x\cos\theta + y\sin\theta). \qquad (3)$$

A quick check: if $\theta = 90^\circ$, we get $x^2 = 4ay$ which is the equation of a parabola with vertex at the origin and its axis along the y-axis. This is as it should be. What happens when $\theta = 180^\circ$? Why?

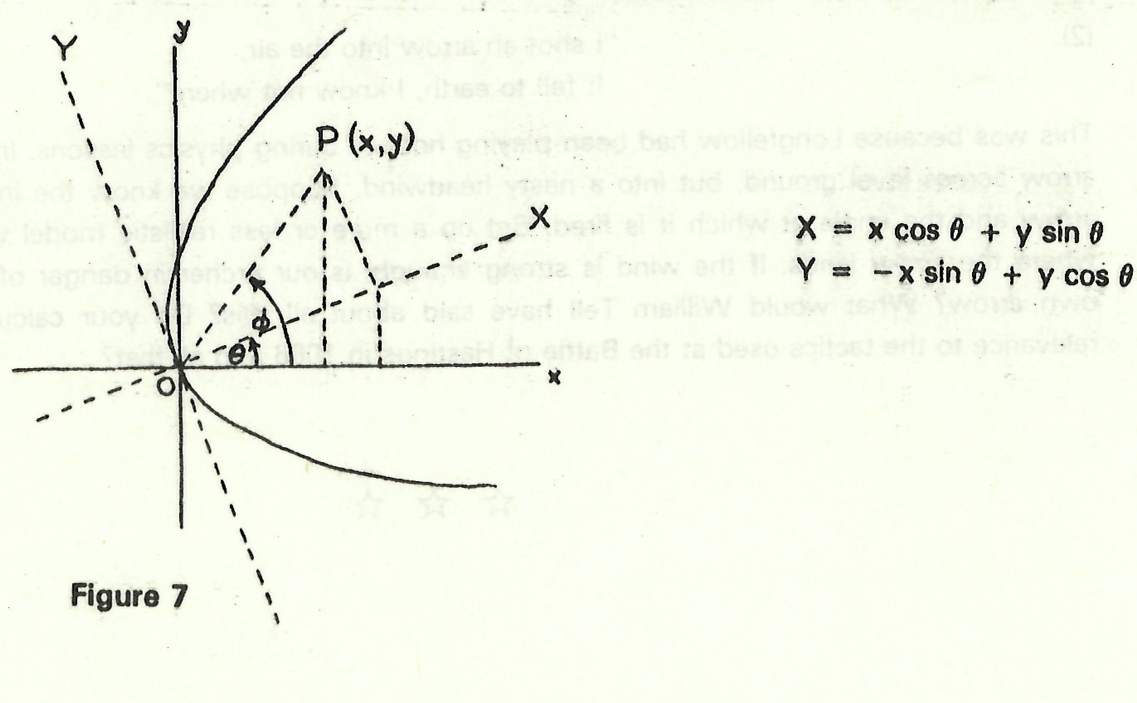

Figure 7

To get the equation of a parabola in its full and most glorious generality, we can first translate the coordinate system, as in (2), and then rotate it, as we did in (3). Thus the equation of a parabola with its vertex at (h,k) and with its axis making an angle θ with the x-axis is

$$\{(y-k)\cos\theta - (x-h)\sin\theta\}^2 = 4a\{(x-h)\cos\theta + (y-k)\sin\theta\}. \qquad (4)$$

The parameter a is just the distance between the vertex and the focus of the parabola. If we multiply out and make some judicious simplifications in the notation, we get an equation of the shape

$$(Ax + By)^2 = Cx + Dy + E. \qquad (5)$$

Does the equation (5) always represent a parabola, whatever values we give the constants A, B, C, D and E? For example, consider the curve $(x+y)^2 = x-y+2$. Can you find its vertex, focal length and the slope of its axis? Try some more examples for luck. If this problem looks too difficult, draw a graph of the curve and use the graph to find the vertex and the slope of the axis of the parabola. This should help you to manoeuvre an equation such as $(x+y)^2 = x-y+2$ back into the form (4) from which you can read off the vital statistics of the parabola. If you have access to a computer, you can use it to help draw graphs of the curves which result from varying A, B, C, D and E in (5). You might come up with an interesting abstract design.

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CURLY PARABOLAS

While we are on the subject of parabolas, here are two more projects.

(1) What is the meaning of the little picture on the cover of this magazine? Why does it work? Is it any use? Who discovered it, and why?

(2)

"I shot an arrow into the air,
It fell to earth, I know not where".

This was because Longfellow had been playing hookey during physics lessons. In fact, he shot his arrow across level ground, but into a nasty headwind. Suppose we know the initial speed of the arrow and the angle at which it is fired. Set up a more or less realistic model which will predict where the arrow lands. If the wind is strong enough, is our archer in danger of being hit by his own arrow? What would William Tell have said about all this? Do your calculations have any relevance to the tactics used at the Battle of Hastings in 1066 and all that?

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