HOW TO AVOID COMMON MATHEMATICAL ERRORS. II.

In Volume 14, Number 2, under this title, we gave you some deliberately faulty "answers" to a number of standard problems. Your corrections and solutions have been decidedly uncommon. Here, we shall set the record straight after first pointing out the moral to our story: the acme of perfection is to Avoid Common Mathematical Errors!

Problems 4, 5, 6, 9, 10, 11 and 14. The errors here all centre round the omission of absolute value signs from the definitions of square roots and logarithms. The correct solutions are as follows.

Problem 9. Draw the graph of $y = \sqrt{(x^2 - 2x + 1)}$.
First simplify: $y = \sqrt{(x-1)^2} = |x-1|$. The graph is as shown in Figure 1.

Problem 10. Draw the graph of $y = \log x^4 - (3/2) \log x^2$.
Simplifying: $y = 4 \log |x| - (3/2)\, 2 \log |x| = \log |x|$. The graph is as shown in Figure 2.

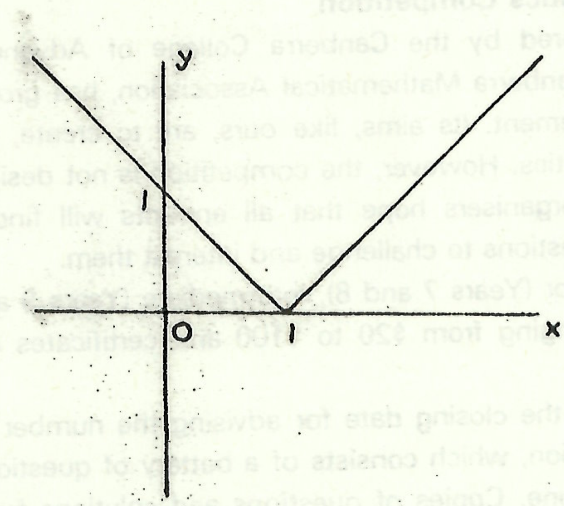

Figure 1

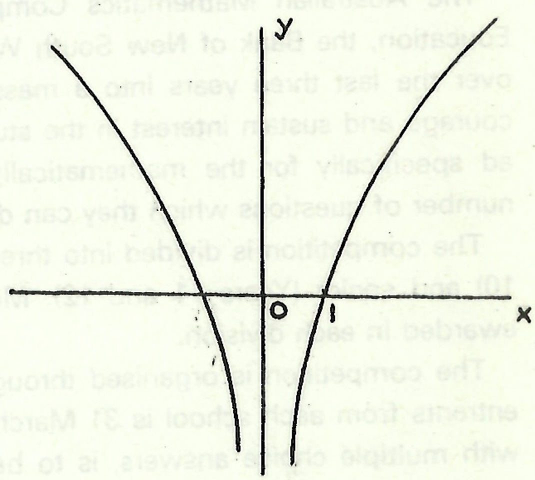

Figure 2

Problem 5. Simplify $\sqrt{(1 + \sin 2x)}$.
We should proceed as follows:

$$\sqrt{(1 + \sin 2x)} = \sqrt{(\sin^2 x + \cos^2 x + 2 \sin x \cos x)} = \sqrt{(\sin x + \cos x)^2} = |\sin x + \cos x|.$$

(It is debatable whether this is any simpler than the original expression.)

Problem 11. Find $\int (ax + b)^{-1}dx$ $(a \neq 0)$.
The correct answer is $\int (ax + b)^{-1}dx = a^{-1} \log |ax + b| + c$.

Problem 4. Find $\int \tan x\, dx$.
Here, $\int \tan x\, dx = \int \sec x \tan x/\sec x\, dx = \log |\sec x| + c$.
(Strictly speaking, we should allow a different constant of integration in each interval $(n - \frac{1}{2})\pi < x < (n + \frac{1}{2})\pi$, $n = 0, \pm 1, \pm 2, \ldots$ A similar observation applies to Problem 11.)

Problem 6. Find the area between the curve $y = 1/x$ and the lines $x = -3$, $x = -1$ and $y = 0$.
Since the curve $y = 1/x$ is below the x-axis for $-3 \leqslant x \leqslant -1$ (see Figure 3), we proceed as follows:

$$\text{Area} = \int_{-3}^{-1} (0 - 1/x)dx = -\int_{-3}^{-1} 1/x\, dx = -\Big[\log |x|\Big]_{-3}^{-1} = -[\log 1 - \log 3] = \log 3.$$

This successfully avoids the use of logarithms of negative numbers.

Problem 14. If $y = x \log x^2$, for what values of x is dy/dx zero?
The correct simplification is $y = 2x \log |x|$ and not $y = 2x \log x$. Consequently, it is easiest to differentiate y in the form given in the problem. We have

$$dy/dx = \log x^2 + x.(1/x^2).2x = 2 \log |x| + 2.$$

So $dy/dx = 0$ if $\log |x| = -1$, that is $x = \pm e^{-1}$. (See Figure 4.)

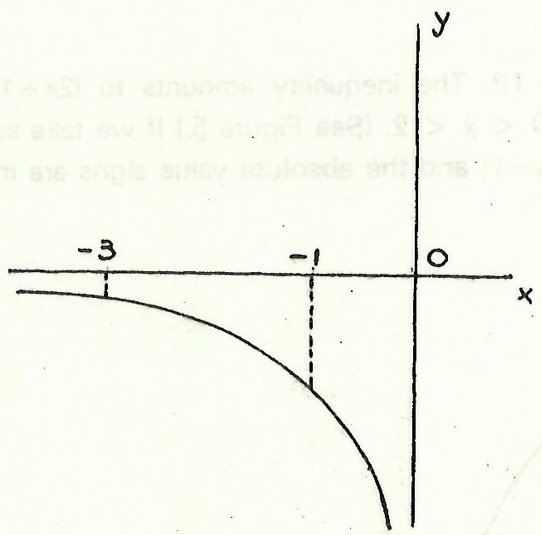

Figure 3

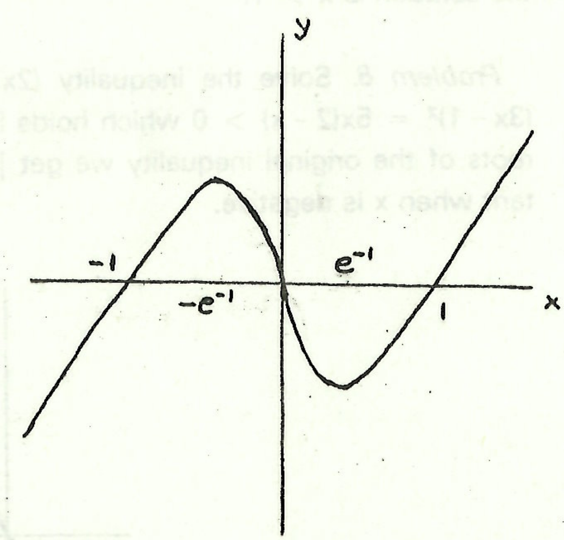

Figure 4

Correct solutions to problems 4, 5, 6, 9, 10 and 11 were received from Surinder Wadhwa (Ashfield Boys' High School).

Problem 1. Solve the equation $\tan 5x = \tan 3x$.
By proceeding mechanically, we readily obtain the solutions $x = \frac{1}{2}\pi n$, $n = 0, \pm 1, \pm 2, \ldots$ But

now we must check by substitution into the original equation to see if all these solutions make sense. If n is even, we do in fact get a solution, but if n is odd we get nonsense. So the correct answer is $x = \pi m$, $m = 0, \pm 1, \pm 2, \ldots$

Problem 2. Solve the equation $2 \sin x + \cos x = -1$.
If we use the formulae $t = \tan \frac{1}{2}x$, $\sin x = 2t/(1+t^2)$, $\cos x = (1-t^2)/(1+t^2)$, we get the solutions $x = -2 \tan^{-1}\frac{1}{2} + 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$ However, we have lost the solutions $x = (2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$ (Check that these values of x do indeed satisfy the original equation.) This happened because the t-substitution does not make sense at the points $x = (2n+1)\pi$, and so these exceptional values need to be checked separately.

Problem 3. Find the equation of the line which passes through the intersection of the lines $x + y - 2 = 0$ and $2x + y + 1 = 0$ and the point $(0, -1)$.
The point of intersection of the lines $x + y - 2 = 0$ and $2x + y + 1 = 0$ is $(-3, 5)$. The line through $(-3, 5)$ and $(0, -1)$ is $2x + y + 1 = 0$. In problems of this sort, if we use the formula $L_1 + \lambda L_2 = 0$ and the given point is on L_2, we will arrive at an impossible equation. It is better to use the form $\lambda L_1 + \mu L_2 = 0$, which always works.

Problem 7. Solve the inequality $\log_x(x+1) > 1$.
Since we are working with a logarithm to base x, we must have $x > 0$. If $0 < x < 1$, then $\log_x(x+1) < 0$. (A neat example here is $x = \frac{1}{2}(\sqrt{5}-1)$, $x+1 = \frac{1}{2}(\sqrt{5}+1) = x^{-1}$, $\log_x(x+1) = -1$). If $x = 1$, then $\log_x(x+1) = \log_1 2$ is undefined. Finally, if $x > 1$, then $\log_x(x+1) > 1$. So the solution is $x > 1$.

Problem 8. Solve the inequality $(2x+1)^2 > (3x-1)^2$. The inequality amounts to $(2x+1)^2 - (3x-1)^2 = 5x(2-x) > 0$ which holds if and only if $0 < x < 2$. (See Figure 5.) If we take square roots of the original inequality we get $|2x+1| > |3x-1|$ and the absolute value signs are important when x is negative.

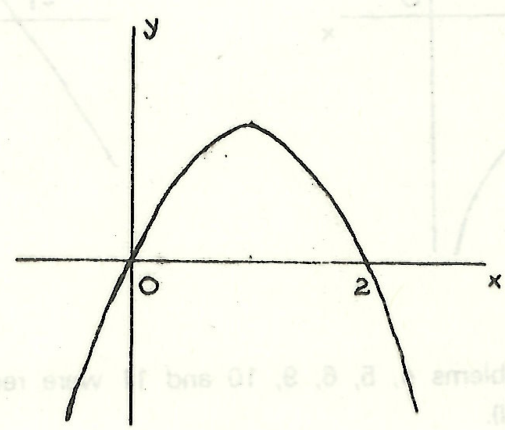

Figure 5

Problem 12. Solve the inequality $|x+1| > -1$.
Clearly, $|x+1| \geqslant 0$ and so the inequality holds for all x. If we first square both sides, we are actually solving the inequality $|x+1| > 1$. (Compare this with Problem 8.)

Problem 13. Solve $3^x . 8^{x/(x+2)} = 6$.
The equation is $3^x . 2^{3x/(x+2)} = 3^1 . 2^1$, that is $3^{x-1} . 2^{(2x-2)/(x+2)} = 1$.
Take logarithms: $(x-1) \log 3 + [(2x-2)/(x+2)] \log 2 = 0$,
that is $(x-1)[\log 3 + 2 \log 2/(x+2)] = 0$.

So there are two solutions: $x = 1$ and $x = -2 \log 2/\log 3 - 2 = -2 \log 6/\log 3$. Check that the second solution works! Equating the exponents of 2 and 3 in the original equation is only valid when all the powers are integers, so that we can appeal to unique factorisation.

—*Michael Hirschhorn*

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Hot from the pages of the Los Angeles Times of 16 November 1978 comes the latest news in the dramatic search for the world's biggest prime number. Curt Noll and Laura Nickel have discovered that the 6533 digit number

$$2^{21701} - 1$$

is a prime. Three years ago, as a project for their school mathematics class, they began to write a computer program to search for large Mersenne primes, that is primes of the form $2^n - 1$. In fact, there are quite a number of primes of this form; $2^n - 1$ is known to be prime for n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 617, 1279, 2203, 2281, 3217, 4219, 4423, 9689, 9941, 11213, 19937, and, now, 21701. When they began their project, this search had gone as far as n = 21000, so they started the prime hunt from there. The computations, performed appropriately in the dead of night, when no-one else wanted the computer, took three months. And they were exceptionally lucky to find a prime so soon at n = 21701. (Look at the gap between the primes occurring at n = 11213 and n = 19937.) Where will the next big prime be? Rumour has it that at this very moment the search goes fearlessly on...

(For a few more details, see Scientific American, January 1979, page 67.)

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