PROBLEM SECTION

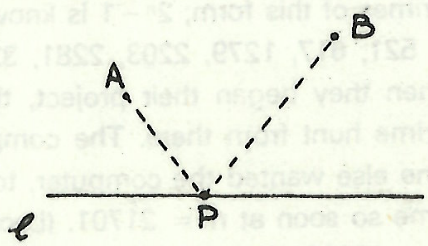
You are warmly invited to submit solutions to one or more of the problems that follow. Please begin each problem on a new page and make sure that each answer bears your name, year and school. Solutions, together with the names of successful solvers, will be published in the issue after next.

Some of the problems are comparatively easy and some are rather hard. If you begin at the beginning, you should find plenty to whet your appetite for the enigmas at the end.

405. If k and N are positive integers with $k > 1$, show that it is possible to find N consecutive odd integers whose sum is N^k.

406. Given n beads numbered 1, 2, 3, . . ., n, show how you can make a single-strand closed necklace from them with the property that the numbers on adjacent beads always differ by either 1 or 2.

407. Let ℓ be a given line and let A and B be two points on the same side of ℓ, as shown in the figure. Find the point P on ℓ with the property that the sum of the distances AP and PB is as small as possible. Prove that your answer is correct.



408. The number 3 can be expressed as the sum of one or more positive integers in 4 ways: 3, 2+1, 1+2 and 1+1+1. Note that the ordering of the summands is significant; 1+2 is counted as well as 2+1. Find a formula for the number of ways in which an arbitrary positive integer n can be so expressed as a sum of positive integers. Prove that your answer is correct.

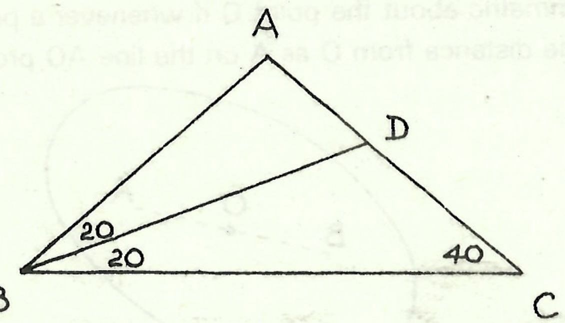
409. In a triangle ABC, BC = 2AC. Produce BA past A to D so that AD = ⅓AB. Prove that CD = 2AD.

410. Mount Zircon is shaped like a perfect cone whose base is a circle of radius 2 miles, and the straight line paths up to the top are all 3 miles long. From a point A at the southernmost point of the base, a path leads to B, a point on the northern slope and 2/5 of the way to the top. If AB is the shortest path on the mountainside joining A to B, find

(i) the length of the whole path AB, and

(ii) the length of the path between P and B, where P is a point on the path at which it is horizontal.

411. The angles ABD, DBC and BCD in the figure are 20°, 20° and 40° respectively. Prove that BC = BD + DA.

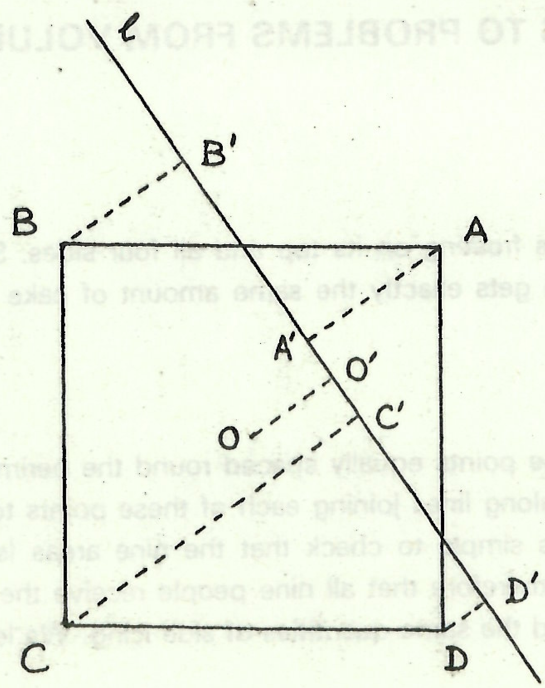


412. Consider a convex polygon with n vertices, and suppose that no three of its diagonals meet at the same point inside the polygon. Determine

(i) the total number of line segments into which the diagonals are divided by their points of intersection, and

(ii) the total number of compartments into which the figure is divided by all its diagonals.

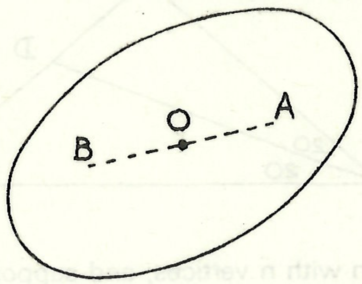
413. In the accompanying figure, O is the centre of the square ABCD and ℓ is a given line. The points O′, A′, B′, C′ and D′ are the feet of the perpendiculars dropped from O, A, B, C and D to the line ℓ. If AA′.CC′ = BB′.DD′ and AB = 2 inches, find OO′. Prove that your answer is correct.



414. Find all positive integers n and k such that the three binomial coefficients ${}^{n}C_{k}$, ${}^{n}C_{k+1}$ and ${}^{n}C_{k+2}$ are in arithmetic progression.

415. Thirty-two counters are placed on a chess-board so that there are four in every row and four in every column. Show that it is always possible to select eight of them so that there is one of the eight in each row and one in each column.

416. Let S be a convex area which is symmetric about the point O. Show that the area of any triangle drawn in S is less than or equal to half the area of S.
(Definition: A set is symmetric about the point O if whenever a point A is in the set, so is the point B which lies at the same distance from O as A on the line AO produced.)



SOLUTIONS TO PROBLEMS FROM VOLUME 14, NUMBER 2

381. A square cake has frosting on its top and all four sides. Show how to cut it to serve nine people so that each one gets exactly the same amount of cake and exactly the same amount of frosting.

Solution:

For example, mark nine points equally spaced round the perimeter of the cake as in the figure. Make nine vertical cuts along lines joining each of these points to O, the centre of the square top surface of the cake. It is simple to check that the nine areas (such as AOB, BOC, COD, in the figure) are all equal and therefore that all nine people receive the same volume of cake, the same quantities of top icing and the same quantities of side icing. We leave it to you to fill in the details.