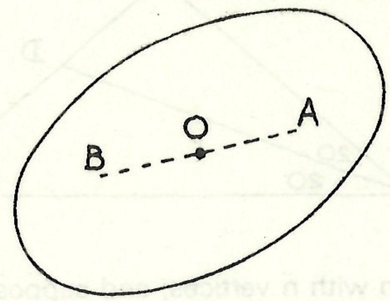
414. Find all positive integers n and k such that the three binomial coefficients ${}^nC_k$, ${}^nC_{k+1}$ and ${}^nC_{k+2}$ are in arithmetic progression.

415. Thirty-two counters are placed on a chess-board so that there are four in every row and four in every column. Show that it is always possible to select eight of them so that there is one of the eight in each row and one in each column.

416. Let S be a convex area which is symmetric about the point O. Show that the area of any triangle drawn in S is less than or equal to half the area of S.
(Definition: A set is symmetric about the point O if whenever a point A is in the set, so is the point B which lies at the same distance from O as A on the line AO produced.)

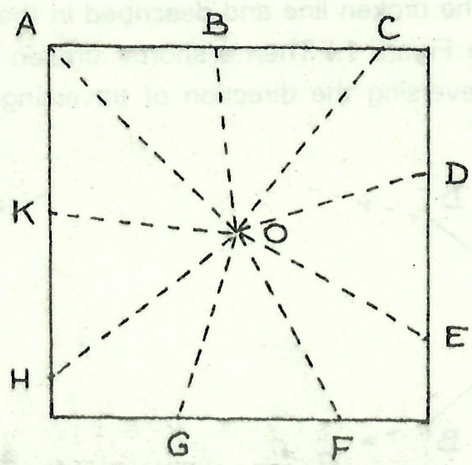


SOLUTIONS TO PROBLEMS FROM VOLUME 14, NUMBER 2

381. A square cake has frosting on its top and all four sides. Show how to cut it to serve nine people so that each one gets exactly the same amount of cake and exactly the same amount of frosting.

Solution:

For example, mark nine points equally spaced round the perimeter of the cake as in the figure. Make nine vertical cuts along lines joining each of these points to O, the centre of the square top surface of the cake. It is simple to check that the nine areas (such as AOB, BOC, COD, in the figure) are all equal and therefore that all nine people receive the same volume of cake, the same quantities of top icing and the same quantities of side icing. We leave it to you to fill in the details.



There are many other solutions. Kurt Svendsen (Busby High School) sent in the following witty answer: First, slice the side icing off two opposite sides of the cake and slice each slab of icing into nine pieces. Then cut the remaining cake into nine slabs. Simple. But this, and also the solution from Peter Crump (Sydney Grammar), have the aesthetic drawback that each person's share is in a number of fragments.

382. Prove or disprove: There are two numbers x,y such that $x + y = 1$, $x^2 + y^2 = 2$ and $x^3 + y^3 = 3$.

Solution from Peter Crump (Sydney Grammar):

Suppose $x + y = 1,$ (1)

$x^2 + y^2 = 2,$ (2)

and $x^3 + y^3 = 3.$ (3)

From (1) and (2), $1 = (x+y)^2 = x^2 + 2xy + y^2 = 2 + 2xy$, hence

$xy = -\frac{1}{2}.$ (4)

From (1) and (3), $1 = (x+y)^3 = x^3 + 3xy(x+y) + y^3 = 3 + 3xy$, hence

$xy = -\frac{2}{3}.$ (5)

Since (4) and (5) are contradictory, there are no numbers x, y satisfying (1), (2) and (3).

Correct solutions were also received from Kurt Svendsen (Busby High School) and Surinder Wadhwa (Ashfield Boys' High School).

383. Let p(1), p(2), . . . p(n) be n points in the plane. Show that the shortest broken line connecting the points does not cross itself.

Solution:

"A broken line connecting the points" means a line made by starting at one of the points, drawing a straight line segment ending at a second point, then another straight line segment from the second point to a third point and so on, until all the points have been reached.

Suppose that such a broken line crosses itself. Let AB and CD be two segments which intersect, occurring in that order in the broken line and described in the sense indicated by the arrows as the broken line is drawn. (See Figure 1.) Then a shorter broken line is obtained by replacing AB and CD by AC and BD and reversing the direction of traversing the line segments joining B to C in Figure 1. (See Figure 2.)

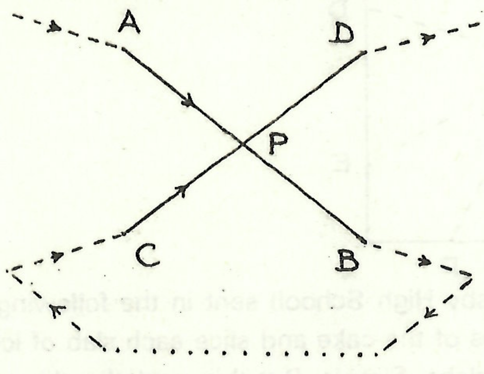


Figure 1

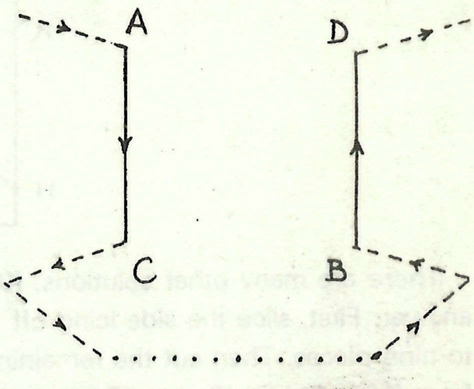


Figure 2

We have to show that the sum of the lengths AC + BD is less than the sum of the lengths AB + CD. Let P be the point of intersection of AB and CD. Then

$$AB + CD = AP + PB + CP + PD = (AP + CP) + (PB + PD) > AC + BD.$$

(Only minor changes in the argument are needed if P coincides with one of A, B, C, or D. If P is itself one of the points, replace the path by AC and BPD to get a shorter line.)

A correct solution was received from Kurt Svendsen (Busby High School.)

384. When the fire alarm went off, the six patrons in the restaurant all hurriedly seized a coat. (1) Safely outside, they discovered that no-one had his own. (2) The coat that Alf had belonged to the man who had seized Bert's. (3) The owner of the coat grabbed by Colin held a coat which belonged to the man who was holding Dave's coat. (4) The man who had seized Ern's coat was not the owner of that grabbed by Fred. Who borrowed Alf's coat? Whose coat did Alf seize? How do you know?

Solution:

A convenient notation for describing the situation is to use "disjoint cycles" as illustrated by, say (ADE)(CF)(B). This would mean that A took D's coat, D took E's and E took A's (completing one "cycle"), that C and F took each other's coats, and B took his own. Notice that each letter occurs exactly once. (We are using A for Alf and so on, so that this particular example does not satisfy the conditions of the problem.) Also, the sum of the "lengths" of the cycles (that is $3+2+1$ in the illustration) is 6.

Now to the problem. By (1), there is no cycle of length 1. By (2), the cycle containing A also contains B with just one letter in between (that is (. . .AxB. . .)). By (3), the cycle containing C also contains D with just two letters in between (that is (. . .CxxD. . .)). If these two cycles are different, we have used 7 letters, and this is not allowed. So A, B, C and D occur in one cycle. There is no way of arranging the 4 letters A, B, C, D to satisfy both requirements, so this cycle is of length at least 5. But it cannot have length 5 since that would leave one letter over to form a cycle of length 1. Hence the actual situation is described by a single cycle of length 6: either (ACBxDx) or (xCxADB), where the letters E and F have yet to be inserted. By (4), F cannot appear two places to the left of E, so the possibilities are (ACBEDF) or (ECFADB). In either case, it was F (Fred) who took A's (Alf's) coat. Alf took either Colin's or Dave's coat.

An excellent solution was received from Gerard Bensoussan (North Sydney Boys' High School).

385. Let v be the number of vertices of a convex polyhedron, e the number of edges, and f the number of faces. Then Euler's formula is $v - e + f = 2$.

(i) Show that for any convex polyhedron $3f \leqslant 2e$ and $3v \leqslant 2e$ (Count the edges round each face, and at each vertex)

(ii) Prove or disprove: It is possible to cut a potato into a convex polyhedron having exactly seven edges.

Solution:

(i) Counting the number of edges round each face and summing gives 2e since each edge has been counted twice, once in each of the two faces which meet along that edge. Thus $2e = af$, where a is the average number of edges per face. Since each face has at least 3 edges, $a \geqslant 3$ and so $2e \geqslant 3f$.

Counting the number of edges at each vertex and summing also gives 2e since each edge has again been counted twice, once at each of the two vertices at its endpoints. Thus $2e = bv$, where b is the average number of edges meeting at a vertex. Since each vertex is the junction of at least 3 edges, $b \geqslant 3$ and so $2e \geqslant 3v$.

(ii) By (i), $3f \leqslant 2.7 = 14$ whence $f \leqslant 4$ and, similarly, $v \leqslant 4$.

Consequently, $v + f - e \leqslant 4 + 4 - 7 = 1$. Thus Euler's formula cannot be satisfied and a convex polyhedron with 7 edges cannot exist.

Alternatively it is obvious that there is no polyhedron with $v \leqslant 3$ and that if $v = 4$, the four vertices must determine a tetrahedron which has only 6 edges.

A correct solution was received from Kurt Svendsen (Busby High School).

386. Determine all polynomials $f(x) = ax^2 + bx + c$ such that

$$f(a) = a, \quad f(b) = b, \quad \text{and } f(c) = c.$$

Solution:

If $a = 0$, the graph of $ax^2 + bx + c$ is a straight line passing through the points (0,0), (b,b) and (c,c). Hence it must be the line $y = x$ unless all of a, b and c are 0. Thus we have so far two possible answers: $f(x) = x$ and $f(x) = 0$.

If $a \neq 0$, the graph of $ax^2 + bx + c$ is a parabola. It cannot pass through three distinct points (a,a), (b,b) and (c,c) which are collinear. Hence we must have either $a = b$, $a = c$, or $b = c$.

First case. Suppose $f(x) = ax^2 + ax + c$ with $a \neq 0$. The condition $f(c) = c$ gives $ac^2 + ac + c = c$, whence $c^2 = -c$, that is $c = 0$ or -1. If $c = 0$, then $f(a) = a$ yields $a^3 + a^2 = a$, whence $a = \frac{1}{2}(-1 \pm \sqrt{5})$. If $c = -1$, then $f(a) = a$ yields $a^3 + a^2 - 1 = a$, that is $(a+1)(a^2-1) = 0$, whence $a = \pm 1$. So we have here two more solutions: $f(x) = a(x^2+x)$ with $a = \frac{1}{2}(11 \pm \sqrt{5})$ and $f(x) = \pm(x^2+x) - 1$.

Second case. Suppose $f(x) = ax^2 + bx + a$ with $a \neq 0$. The condition $f(a) = a$ gives $a^3 + ab + a = a$, whence $b = -a^2$. The condition $f(b) = b$ gives $ab^2 + b^2 + a = b$ which, with our previous conclusion, gives $a^5 + a^4 + a^2 + a = 0$, that is $a(a+1)^2(a^2-a+1) = 0$. If $a = -1$, then $b = -1$ giving a solution already obtained in the first case. If $a^2 - a + 1 = 0$, we obtain a complex solution: $a = c = \frac{1}{2}(1 \pm \sqrt{(-3)})$, $b = \frac{1}{2}(-1 \pm \sqrt{(-3)})$ and $f(x) = ax^2 - a^2x + a$.

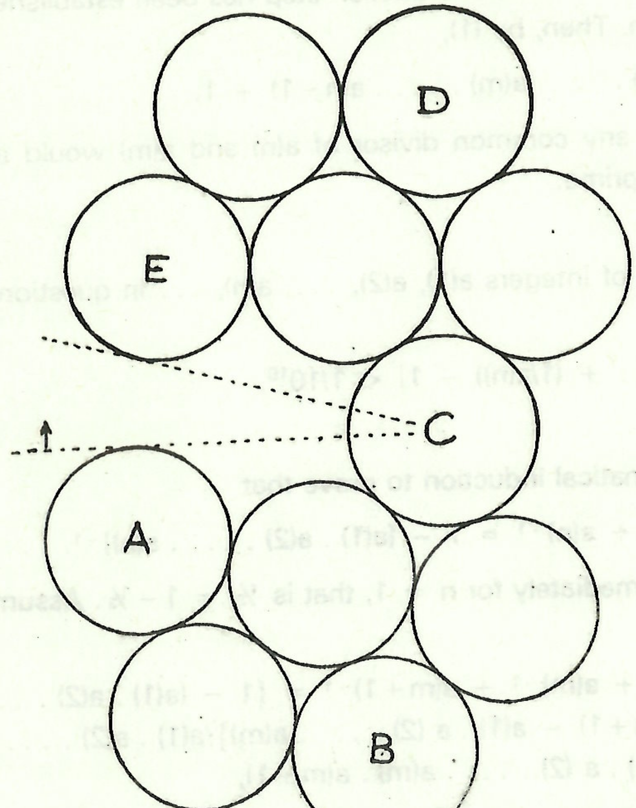
Third case. Suppose $f(x) = ax^2 + bx + b$ with $a \neq 0$. The condition $f(b) = b$ gives $ab^2 + b^2 + b = b$, that is $(a+1)b^2 = 0$, so $a = -1$ or $b = 0$. If $a = -1$, then $f(a) = a$ yields $a^3 + ab + b = a$, that is $-1 - b + b = -1$ which is true for all b. If $b = 0$, the condition $f(a) = a$ gives $a^3 = a$, whence $a = \pm 1$. So we have the final two solutions: $f(x) = -x^2 + bx + b$ for any b and $f(x) = \pm x^2$.

Partly correct solutions were received from Peter Crump (Sydney Grammar), Surinder Wadhwa (Ashfield Boys' High School) and Otis Wright (Davidson High School), all of whom managed to "lose" some solutions.

387. Finitely many pennies are placed on a flat surface, no two overlapping. Prove or disprove: No matter how this is done, it is always possible to paint each penny with one of three colours so that no two pennies having the same colour touch each other.

Solution.

The statement is false. Consider the arrangement of coins in the figure. If we commence colouring the coins using the colours red, white and blue by painting A red, then it is easy to see that the coins B, C, D and E must also be red. Now pivot the lower arm of 6 coins from A to C about the centre of C until A makes contact with E. One or other of these two coins now requires a new colour.



The counterexample is rather surprising. Can you find an arrangement with fewer than 11 coins which disproves the assertion in the problem?

388. Let a list of integers a(1), a(2), . . ., a(n), . . . be defined in succession by

$a(n+1) = (a(n))^2 - a(n) + 1$ and $a(1) = 2$.

The first few are $a(1) = 2$, $a(2) = 3$, $a(3) = 7$, $a(4) = 43$, $a(5) = 1807, \ldots$

Show that the integers a(1), a(2), a(3), . . . are pairwise relatively prime (i.e. if a(k) and a(ℓ) are any two different members of the list, they have no common factor except 1).

Solution:

We prove first, using mathematical induction, that

$$a(n) = a(1) \,.\, \ldots \,.\, a(n-1) + 1 \text{ for } n \geqslant 2, \qquad (1)$$

that is, each a(n) is one more than the product of all previous terms in the list. This can be checked immediately for n = 2. Now suppose (1) is true for n = m, say, and consider a(m+1). From the definition of the sequence,

$$\begin{aligned} a(m+1) &= a(m)^2 - a(m) + 1 = [a(m)-1]a(m) + 1 \\ &= [a(1) \,.\, a(2) \,.\, \ldots \,.\, a(m-1)]a(m) + 1. \end{aligned}$$

This is (1) with $n = m+1$. Thus the induction step has been established and (1) is proved.

Now suppose $n > m$. Then, by (1),

$$a(n) = a(1) \,.\, a(2) \,.\, \ldots \,.\, a(m) \,.\, \ldots \,.\, a(n-1) + 1,$$

whence it is clear that any common divisor of $a(n)$ and $a(m)$ would also divide 1. Thus $a(n)$ and $a(m)$ must be relatively prime.

389. For the same list of integers $a(1), a(2), \ldots, a(n), \ldots$ in question 388 show that by taking n very large

$$|(1/a(1)) + (1/a(2)) + \ldots + (1/a(n)) - 1| < 1/10^{10}.$$

Solution:

We again use mathematical induction to prove that

$$a(1)^{-1} + a(2)^{-1} + \ldots + a(n)^{-1} = 1 - [a(1) \,.\, a(2) \,.\, \ldots \,.\, a(n)]^{-1}. \qquad (2)$$

This can be checked immediately for $n = 1$, that is $\tfrac{1}{2} = 1 - \tfrac{1}{2}$. Assume that (2) is true for $n = m$. Then

$$\begin{aligned} a(1)^{-1} + a(2)^{-1} + \ldots + a(m)^{-1} + a(m+1)^{-1} &= \{1 - [a(1) \,.\, a(2) \,.\, \ldots \,.\, a(m)]^{-1}\} + a(m+1)^{-1} \\ &= 1 - \{a(m+1) - a(1) \,.\, a(2) \,.\, \ldots \,.\, a(m)\}/a(1) \,.\, a(2) \,.\, \ldots \,.\, a(m)\, a(m+1) \\ &= 1 - 1/a(1) \,.\, a(2) \,.\, \ldots \,.\, a(m) \,.\, a(m+1), \end{aligned}$$

where we have used (1) in the answer to question 388 to make the last step. This gives (2) for $n = m+1$, whence (2) must be true for all n.

Thus the sum of reciprocals on the left of (2) differs from 1 by $[a(1) \,.\, a(2) \,.\, \ldots \,.\, a(n)]^{-1}$ which is less than 10^{-10} for all $n \geqslant 6$.

390. Let $b(1), b(2), \ldots, b(n)$ be any positive numbers

Prove that $(b(1) + b(2) + \ldots + b(n))((1/b(1)) + (1/b(2)) + \ldots + (1/b(n)) \geqslant n^2$

Solution:

If x and y are both positive, then

$$x/y + y/x \geqslant 2. \qquad (1)$$

This follows from $(x-y)^2 \geqslant 0$, that is $x^2 + y^2 \geqslant 2xy$, after dividing through by xy.

Now

$$\begin{aligned} S &= [b(1) + b(2) + \ldots + b(n)][b(1)^{-1} + b(2)^{-1} + \ldots + b(n)^{-1}] \\ &= \sum_{1 \leqslant j \leqslant n} b(j)/b(j) + \sum_{1 \leqslant i < j \leqslant n} \{b(i)/b(j) + b(j)/b(i)\}. \end{aligned}$$

The first part is the sum of n terms each equal to 1. The second part is the sum of $\tfrac{1}{2}n(n-1)$ terms (that is the number of ways of choosing two different numbers i and j from $\{1, 2, \ldots, n\}$) and each of these terms is greater than or equal to 2, in view of (1). Hence

$$S \geqslant n.1 + \tfrac{1}{2}n(n-1).2 = n^2.$$

391. Each of three classes has n students. Each student knows altogether $(n+1)$ students in the other two classes. Prove that it is possible to select one student from each class so that all three know one another. (Acquaintances are always mutual).

Solution:

Let k (≥ 1) be the smallest number of acquaintances of any student with the students in one of the other classes. Label the classes A, B and C, so that there is a student a in class A who has exactly k acquaintances in class B and let b be one of these acquaintances. Now, a has $(n+1)-k$ acquaintances in class C, so there are only $k-1$ students in C not acquainted with a. By our definition of k, b knows at least k students in C and at least one of these, c say, must be known to a. Thus a, b and c are mutual acquaintances from the three classes.

A partial solution was received from Surinder Wadhwa (Ashfield Boys' High School).

392. Let S consist of the set of all points (x,y) in the Cartesian plane such that x and y are both integers. The centre of gravity of the triangle with vertices $(x(1),y(1))$, $(x(2),y(2))$, $(x(3),y(3))$ is the point $((x(1) + x(2) + x(3))/3, (y(1) + y(2) + y(3))/3)$.

Prove that out of any 9 points in S, it is always possible to choose 3 with the property that the centre of gravity of the triangle formed by them is also a point in S.

Solution:

We have to show that given any nine points $(x(i), y(i))$ in S, it is possible to find three of them, say $i = k$, ℓ and m, so that $x(k)+x(\ell)+x(m)$ and $y(k)+y(\ell)+y(m)$ are both multiples of 3. Since only remainders on division by 3 are relevant, we represent the points by crosses on the grid in Figure 1; for example, if we have a point whose x-coordinate has remainder 1 mod 3 and whose y-coordinate has remainder 2 mod 3, then we put a cross in the (1,2)-square in Figure 1, as shown.

Now, observe that the sum of three integers a, b and c can give a multiple of 3 in only two ways: either a, b and c all have the same remainder mod 3, or these remainders are some arrangement of 0, 1 and 2 (each occurring once). If there are three points in some square of the grid in Figure 1 then the x-coordinates of these points all have the same remainder mod 3 and their y-coordinates all have the same remainder mod 3, so these 3 points have the required property. The only other way to achieve this end is to find three points which lie in the same row or column of the grid (Figure 2), or three points forming a pattern with just one in each row and one in each column (Figure 3).

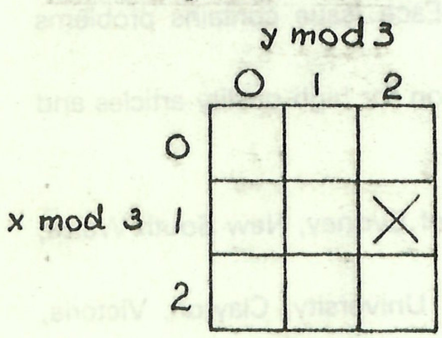


Figure 1

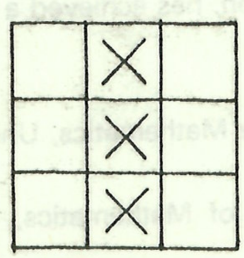


Figure 2

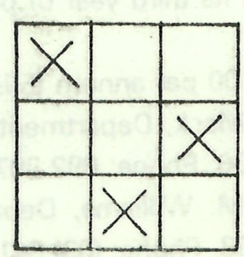


Figure 3

From the above remarks, we can suppose that at most two of the points fall in any square of the grid; since there are nine points in all, at least five squares in the grid are occupied. Also, we can suppose that at most two squares in any row or column are occupied, because if we have three points in a row or column (as in Figure 2), these points solve the problem. We have to show that if we draw five crosses on our grid with at most two in any row or column, then we can choose three forming the pattern typified by Figure 3.

Under the above assumptions, there are two rows each with two crosses and one row with just one cross. Since the labels in Figure 1 are no longer relevant, we can suppose the first row is the one with a single cross. Similarly, there is one column containing a single cross. If this special column is the first column, then the five crosses must be as in Figure 4 and the points corresponding to the bold crosses solve the problem. Otherwise, we can suppose the special column is the second, and then we get Figure 5 which also has three points with the required property.

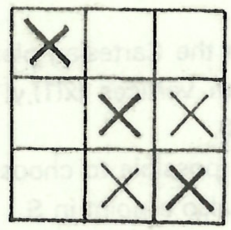


Figure 4

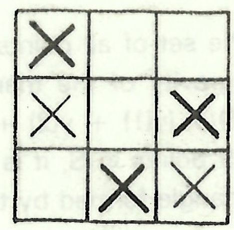


Figure 5

A correct solution was received from Kurt Svendsen (Busby High School).

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