LETTERS TO THE EDITOR

PI'S THE LIMIT

Dear Sir,

Before Professor Prokhovnik's article, "The geometric nature of π", appeared (Parabola, Volume 14, Number 3), I had calculated the values of the area and perimeter of regular polygons circumscribed and inscribed round a circle. For a regular n-gon circumscribed about a circle of radius r, the area of the polygon is $nr^2 \tan(180°/n)$ and the perimeter is $2nr \tan(180°/n)$. These both give $n \tan(180°/n) \to \pi$ as $n \to \infty$. (This is the construction dealt with in Professor Prokhovnik's article.)

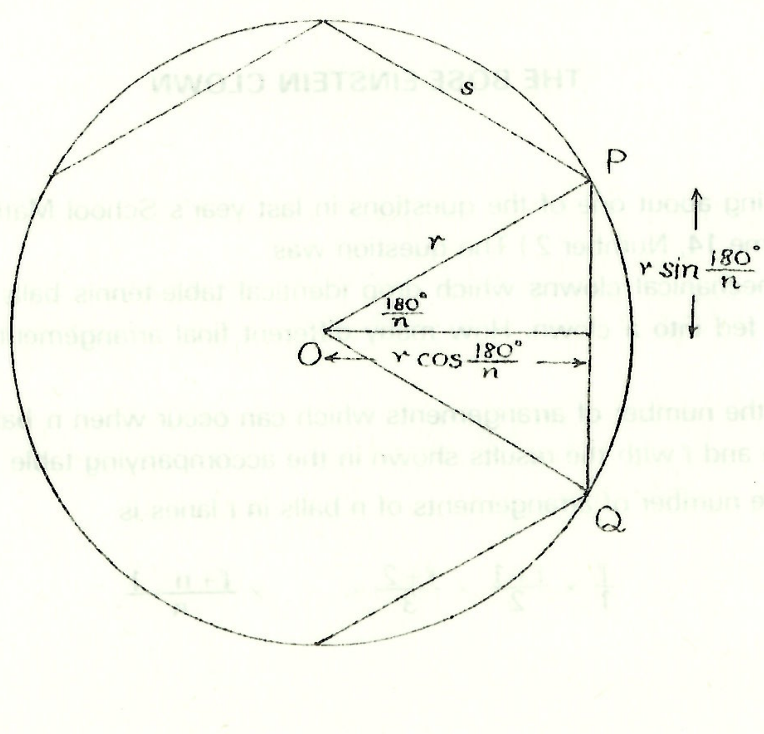
Now consider a regular n-gon inscribed in a circle of radius r. From the figure, the length, s, of a side of the polygon is $s = 2r \sin(180°/n)$, so the perimeter of the polygon is $2nr \sin(180°/n)$. As n gets larger, the polygon approaches a circle with circumference $2\pi r$, so

$$n \sin(180°/n) \to \pi \text{ as } n \to \infty. \qquad (1)$$

Also, from the figure, the altitude of triangle OPQ is $r \cos(180°/n)$, so the area of this triangle is $r^2 \sin(180°/n) \cos(180°/n)$. Hence the area of the polygon is $nr^2 \sin(180°/n) \cos(180°/n)$ and this approaches the area πr^2 of the circle as n gets larger. So

$$n \sin(180°/n) \cos(180°/n) \to \pi \text{ as } n \to \infty.$$

Actually this last limit follows from (1) since $\cos(180°/n)$ approaches 1 when n is large.



Thus, not only does n tan (180°/n) approach π, but n sin (180°/n) does also. Actually for any value of n, n sin (180°/n) is closer to π than n tan (180°/n) is.

Richard Wilson,
Year 10,
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Editor's comments.

We can draw another conclusion from Richard's calculations. Observe that the perimeter of the inscribed polygon is shorter than the circumference of the circle, while the perimeter of the circumscribed polygon is longer. So, after dividing by 2r, we have

$$n \sin (180^\circ/n) < \pi < n \tan (180^\circ/n). \qquad (2)$$

For example, if we take n = 6 here, we get $3 < \pi < 2\sqrt{3} \approx 3.46$. Both of the inequalities in (2) are essential for Archimedes' determination of π, since they give not only an approximate value for π, but also an estimate of the error. Archimedes' result for π was

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

We can get this from (2) by repeatedly doubling n and using double-angle formulae to calculate the trigonometric ratios at each step. If $n = 96 = 6 \times 2^4$, then (2) gives

$$3.1410\ldots < \pi < 3.1427\ldots$$

and this is sufficiently accurate to give Archimedes' inequalities. We would need 4 applications of the double-angle formulae to get the result.

It is interesting that, as Richard observes, the lower estimate in (2) is better than the upper one. In fact, for large n, the error in approximating π by n sin (180°/n) is about half the error in approximating by n tan (180°/n). Can any of our readers explain this?

THE BOSE-EINSTEIN CLOWN

Dear Sir,

I have been thinking about one of the questions in last year's School Mathematics Competition. (See Parabola, Volume 14, Number 2.) The question was:

"A sideshow has mechanical clowns which drop identical table-tennis balls onto boards with six alleys. Six balls are fed into a clown. How many different final arrangements of the balls can occur?"

I have calculated the number of arrangements which can occur when n balls are fed into ℓ lanes for small values of n and ℓ with the results shown in the accompanying table.

I conclude that the number of arrangements of n balls in ℓ lanes is

$$\frac{\ell}{1} \times \frac{\ell+1}{2} \times \frac{\ell+2}{3} \times \ldots \times \frac{\ell+n-1}{n}.$$

ℓ \ n	1	2	3	4	5	6	n
1	1	1	1	1	1	1	$\frac{1}{1} \times \frac{2}{2} \times \frac{3}{3} \times \ldots \times \frac{n}{n}$
2	2	3	4	5	6	7	$\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \ldots \times \frac{n+1}{n}$
3	3	6	10	15	21	28	$\frac{3}{1} \times \frac{4}{2} \times \frac{5}{3} \times \ldots \times \frac{n+2}{n}$
4	4	10	20	35	56	84	$\frac{4}{1} \times \frac{5}{2} \times \frac{6}{3} \times \ldots \times \frac{n+3}{n}$
5	5	15	35	70	126	210	$\frac{5}{1} \times \frac{6}{2} \times \frac{7}{3} \times \ldots \times \frac{n+4}{n}$
6	6	21	56	126	252	462	$\frac{6}{1} \times \frac{7}{2} \times \frac{8}{3} \times \ldots \times \frac{n+5}{n}$

Can you explain how this formula comes about?

Alan Ridout,
Year 12,
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Editor's comments.

According to Alan's formula, the number of ways of putting n identical objects in ℓ boxes is equal to the binomial coefficient

$$^{\ell+n-1}C_n = (\ell+n-1)!/n!(\ell-1)!$$

On the other hand, this binomial coefficient also tells us the number of ways of arranging $\ell+n-1$ objects in a line if there are n indistinguishable objects of one sort and $\ell-1$ indistinguishable objects of another sort. (To see this, observe that there are $(\ell+n-1)!$ arrangements of $\ell+n-1$ objects in a line. But we can then permute the n objects of the first kind among themselves in n! ways and we can permute the $\ell-1$ objects of the second kind in $(\ell-1)!$ ways and these permutations leave the pattern of the $\ell+n-1$ objects unchanged. So we have $(\ell+n-1)!/n!(\ell-1)!$ different arrangements of the $\ell+n-1$ objects.) The problem now is to see why these two apparently different counting problems lead to the same answer.

Suppose, in the second problem, that we have $n = 6$ circles and $\ell - 1 = 5$ lines to be arranged in a row. Here is a typical arrangement:

O | | OO | | | OOO

We can reinterpret the $\ell-1 = 5$ lines as fences between $\ell = 6$ boxes. That is, we have an arrangement of $n = 6$ objects in $\ell = 6$ boxes, the boxes containing 1, 0, 2, 0, 0, 3 objects respectively. This argument shows that the two problems are indeed counting the same thing.

This trick is worth remembering, and not just for solving problems in the School Mathematics Competition. It is sometimes referred to as Bose-Einstein statistics because it is the sort of counting performed by protons, neutrons and similar particles in fulfilment of the laws of quantum mechanics. Electrons, on the other hand, count with Fermi-Dirac statistics. We can turn our clown into a Fermi-Dirac clown if we arrange things so that at most one ball can go into each alley. Can you figure out how to count the number of arrangements in this case? Can you find out where all these funny names came from and what they mean?

A FRIEND AT THE AIRPORT?

Dear Sir,

I have found some more amusing facts concerning palindromes. I construct a modified Pascal's triangle as in the figure. To get each new entry in the triangle, add the two numbers above and immediately to the left and right as usual, but if this gives a two-digit number, add the digits together and write this answer into the triangle. For example, to get the third entry in the fifth row, add 4 + 6 = 10, but instead of writing down 10, add again 1 + 0 = 1 and write down 1 in the triangle.

* 0	1
* 1	1 1
* 2	1 2 1
* 3	1 3 3 1
4	1 4 6 4 1
* 5	1 5 1 1 5 1
6	1 6 6 2 6 6 1
7	1 7 3 8 8 3 7 1
* 8	1 8 1 2 7 2 1 8 1
9	1 9 9 3 9 9 3 9 9 1
10	1 1 9 3 3 9 3 3 9 1 1
*11	1 2 1 3 6 3 3 6 3 1 2 1
12	1 3 3 4 9 9 6 9 9 4 3 3 1
13	1 4 6 7 4 9 6 6 9 4 7 6 4 1
14	1 5 1 4 2 4 6 3 6 4 2 4 1 5 1
15	1 6 6 5 6 6 1 9 9 1 6 6 5 6 6 1
16	1 7 3 2 2 3 7 1 9 1 7 3 2 2 3 7 1
*17	1 8 1 5 4 5 1 8 1 1 8 1 5 4 5 1 8 1
18	1 9 9 6 9 9 6 9 9 2 9 9 6 9 9 6 9 9 1

I noticed that when the number formed by one of the rows marked * is multiplied by 11, the answer is the number in the next row.

Could you explain this?

Paul Rider,
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